The probability of choosing the unknot among 2-bridge knots

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Joint with Sunder Ram Krishnan

Technion - Israel Institute of Technology

Glances Manifolds, Jagiellonian University, July 17th, 2015

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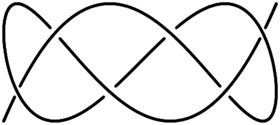

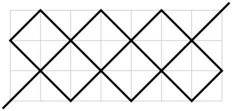

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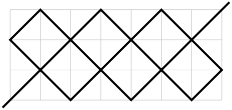
Overview

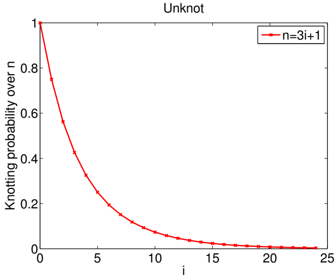
Given a billiard table diagram $T(3, n+1)$,

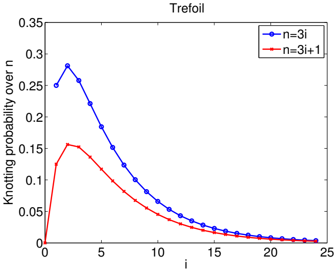


choose its n crossings uniformly at random.

We give an exact formula for the probability of obtaining a knot.

Overview


Unknot
n=3i+1
Knotting probability over n
i
0
0.2
0.4
0.6
0.8
1
5
10
15
20
25


Trefoil
n=3i
n=3i+1
Knotting probability over n
i
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
5
10
15
20
25

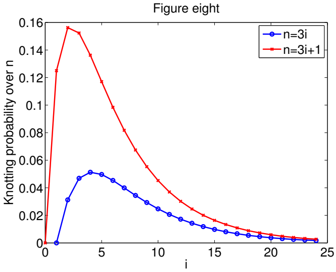
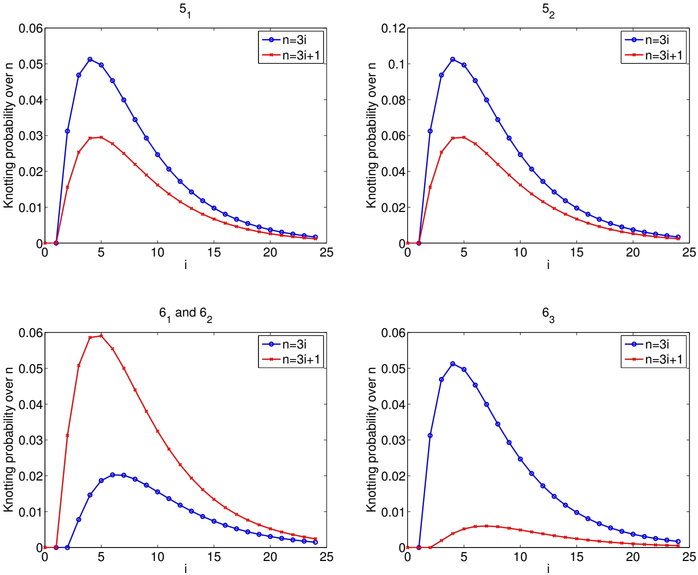


Figure eight
n=3i
n=3i+1
Knotting probability over n
i
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
5
10
15
20
25

Overview



5_1
5_2
6_1 and 6_2
6_3
Knotting probability over n
i
n=3i
n=3i+1
0.06
0.05
0.04
0.03
0.02
0.01
0
0.12
0.1
0.08
0.06
0.04
0.02
0
5
10
15
20
25

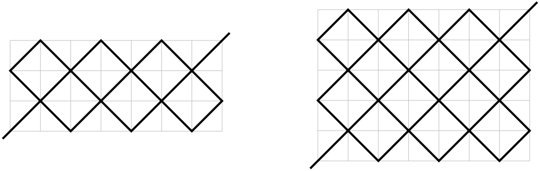
Outline

I. Motivation and other random knot models

II. Chebyshev knots $T(a, b)$

- ***[Koseleff-Pecker '11]***
  Present any k-bridge knot by $T(a, b)$ for $a = 2k - 1$

- ***[C. '14]*** Jones polynomials of $T(3, b)$ and $T(5, b)$



III. Random billard table diagrams $T(3, b)$

- ***[C.-Krishnan '15]*** Probability of any 2-bridge knot

The ***probabilistic method*** has been used in graph theory to try to understand very large graphs.

- ***[Erdös - Rényi]*** random graph model

This method has seen some early results in topology.

- ***[Linial-Meshulam '06]*** random k-complexes
- ***[Costa-Farber '14]*** generalized random k-cxes
- ***[Dunfield-Thurston(s) '06]*** random 3-manifolds

The goal is to find evidence to support or refute the following.

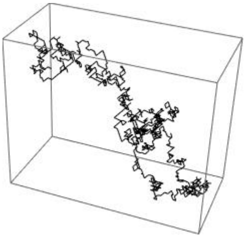
Conjecture:

The Jones polynomial detects the unknot.

I. Random knot models from the literature / Physical

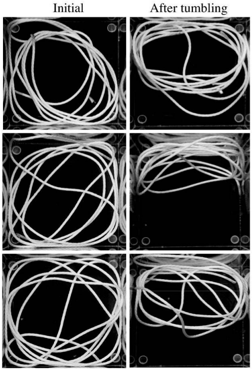
- ***[AMS Special Session in Vancouver '93]***
  and ***[Series on Knots and Everything '94]***

- ***[Dobay-Sottas-Dubochet-Stasiak '01,'02]*** random walk in $\mathbb{Z}^3$



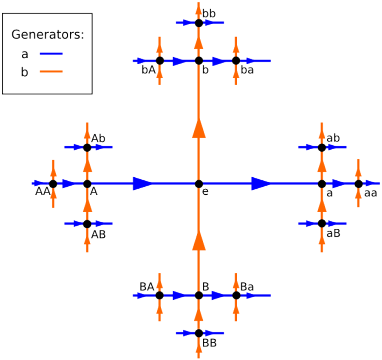
- Slightly different settings by, e.g., ***[Buck (Gregory) '94]***
  [Deguchi-Tsurusaki '94], ***[Aliashvili '05]***
  [Arsuaga-Blackstone-Diao-Hinson-Karadayi-Saito '07]

I. Random knot models from the literature / Physical



[Raymer-Smith ’07] “Spontaneous knotting of an agitated string”

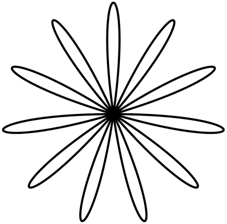
Random walks on Cayley graphs

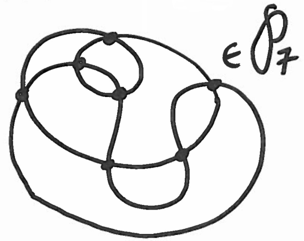


- e.g. ***[Nechaev-Grosberg-Vershik '96]***, ***[Voituriez '02]***, ***[Nechaev-Voituriez '03]***, ***[Mairesse-Matheus '07]***

I. New random knot diagrams

- 2011 grant proposal ***[C.-Linial-Nowik]***
- ***[Even-Zohar–Hass–Linial–Nowik ’14]*** random Petaluma knots





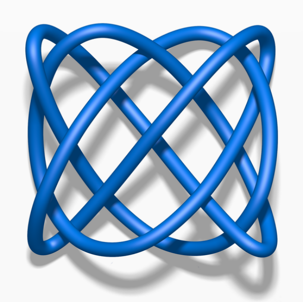
- ***[Dunfield-Obeidin-et al]*** from random 4-valent graphs
- ***[Babson-Westenberger]*** random projections of high-dimensional knots

II. Parametrizations

A ***Lissajous knot*** is of the form

$$x = \cos(\eta_x t + \phi_x),$$
$$y = \cos(\eta_y t + \phi_y),$$
$$z = \cos(\eta_z t + \phi_z),$$

where $t, \phi_i \in \mathbb{R}$,
$\eta_i \in \mathbb{Z}$ are coprime.



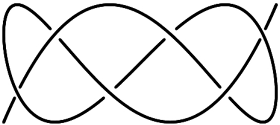
Not all knots are Lissajous, however,
e.g. torus knots and the figure eight.

Studied by, e.g., ***[Bogle-Hearst-Jones-Stoilov '94]***, ***[Jones-Przytycki '98]***, ***[Przytycki '98]***.

A ***Chebyshev knot*** is $x = T_a(t)$, $y = T_b(t)$, $z = T_c(t+\varphi)$,

where $t \in \mathbb{R}$, $a, b, c \in \mathbb{Z}$ are coprime, $\varphi \in \mathbb{R}$ a constant, and

$T_n(\cos t') = \cos(nt')$ is the n-th Chebyshev poly (first kind).



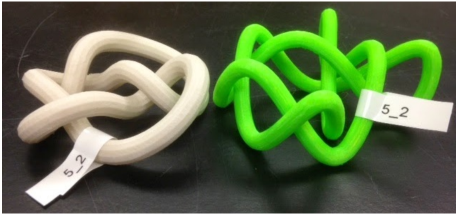
Proposition [Koseleff-Pecker '11]:

All knots are Chebyshev. (More on this later.)

A ***billiard knot*** is the trajectory of a ball traveling in a 3D domain at a straight line, reflecting perfectly off the walls at rational ∠.

Proposition [Jones-Przytycki '98]:

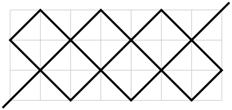
Lissajous knots are precisely the billiard knots in a cube.

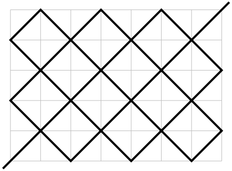


MakerHome: One 3D print every day from home, for a year

Monday, May 26, 2014: Day 273 - Lissajous conformation of 5_2.

Chebyshev knots can be projected onto
billiard table diagrams T(a,b).





To simplify, we replace $c \in \mathbb{N}$ and $\phi \in \mathbb{R}$ with a string of $+, -$

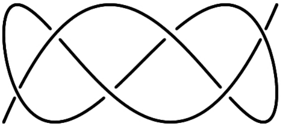
of length N corresponding to the $N = \frac{1}{2}(a-1)(b-1)$ crossings.

II. Natural indexing by bridge number

Proposition [Koseleff-Pecker ’11]:

For knot K and $br(K) \leq m \in \mathbb{N}$, K is some $T(a, b)$

where $a = 2m - 1$ and $b \equiv 2 \pmod{2a}$.



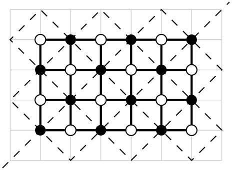
Theorem [Koseleff-Pecker ’11]:

Every knot has a projection that is a Chebyshev plane curve.

II. Alexander polynomial via grid graphs

The ***"dimer" graph*** ***[C.-Dasbach-Russell '14]*** for a billiard table diagram is the popular ***grid graph*** from graph theory.

Dimer or perfect matching models on these grid graphs also appear in statistical mechanics.



The ***Alexander polynomial*** can be swiftly computed from this ***[C.-Dasbach-Russell '14]***.

Consider $T(a, b)$ with $a = 3$ or $a = 5$ and with b coprime.

Order the N crossings lexicographically.

Obtain a knot from a string of $\{+, -\}$ of length N.

Goal [C. '14]:

To compute the Jones polynomials directly from the string.

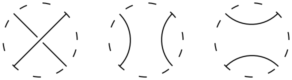
Want a notation that is sensitive to whether a crossing is $\pm$.

Let f_b be the ***Kauffman bracket polynomial*** $\langle T(3, b) \rangle$

We compute writhe at the end.

Apply the ***unoriented Skein relation***

$$\langle L \rangle = A\langle L_0 \rangle + A^{-1}\langle L_\infty \rangle$$



Notation:

To each crossing assign some monomial:

If the $\pm$ crossing is smoothed vertically, use $A^{\pm}$.

If the $\pm$ crossing is smoothed horizontally, use $A^{\mp}$.

If the $\pm$ crossing is resolved by Reidemeister I, use $f_2^{\pm} = -A^{\mp 3}$

Let $C = [A^{\pm}, A^{\pm}] + [f_2^{\mp}, A^{\mp}]$ and substitute C from the left.

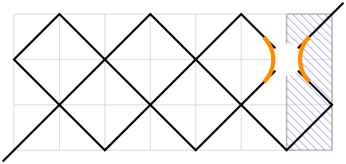
Theorem [C.]:

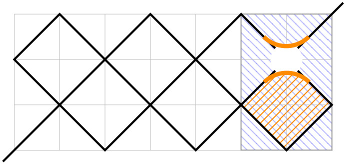
The Kauffman bracket polynomials f_b of $T(3, b) \supseteq$ {2-bridge knots} obey the following recursion rules:

If a summand in f_{b-1} ends in _____ then it is a summand in f_b:

$(..., A^{\pm})$	with a C replacing the $A^{\pm}$
$(..., [f_2^{\mp}, A^{\mp}])$	ending with $A^{\pm}$
$(..., C)$	with $[C, A^{\pm}] + [A^{\pm}, f_2^{\mp}, A^{\mp}]$ replacing the C.

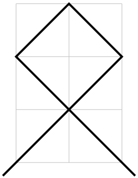
II. Jones polynomials of 2-bridge knots / Proof Idea



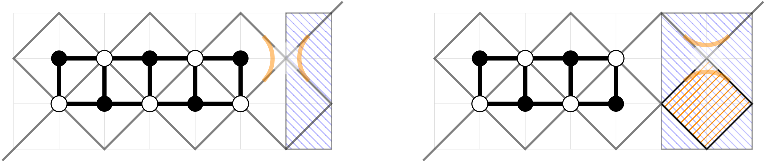


$(\ldots, A^{\pm})$ $(\ldots, f_2^{\mp}, A^{\mp})$

where $f_2^{\pm} = -A^{\mp 3}$ is the Kauffman bracket polynomial of $T(3,2)$



Look at the “dimer graph” again to see the $2 \times b$ grid graph.



The smoothings relate to the $2 \times (b - 1)$ and $2 \times (b - 2)$ grid graphs.

Proposition [C.]:

The # of terms in the expansion of f_b is
(a sequence that is an offset by four of)

$$p(0) = 1,$$

$$p(1) = a(2) = 0,$$

$$p(n) = p(n-2) + p(n-3),$$

the ***Padovan sequence***,

$[A000931]$ in The On-Line Encyclopedia of Integer Sequences.

We now apply the probabilistic method to this model.

Take $T(3, b)$ for fixed b with a random string in $\{+, -\}^{b-1}$.

Compare among all 2-bridge knots together with the unknot.

Goal:

Prove robustness of this model;

generalize for larger bridge numbers.

Main Theorem [C.-Krishnan '15]:

The probability of a given knot appearing in $T(3, n+1)$ is

$$\frac{1}{2^n}\left(\mathcal{X}_{n+3} + \sum_{i=0}^{n-6}\left[\binom{n-5}{i+1} + 4\binom{n-5}{i}\right]\mathcal{X}_{n-3i} + 4\mathcal{X}_{-2n+15}\right).$$

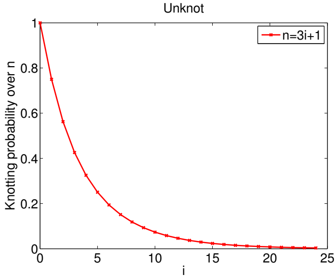
For a knot K with reduced lengths $\ell_0 \equiv 0$ and $\ell_1 \equiv 1 \mod 3$ and $\#(\ell)$ the number of ways to write K in that length,

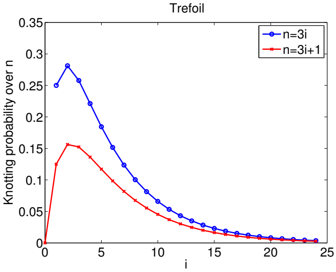
$$\mathcal{X}_n = 4(m-1)\#(\ell) \text{ for } n = 3m + \ell \text{ and } m \geq 2$$

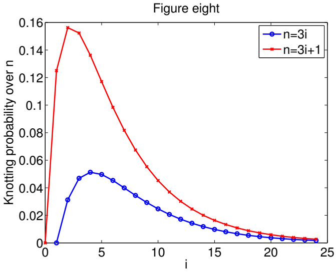
For $n \geq 0$ and $N = \left\lfloor \frac{n-\ell}{2} \right\rfloor + \ell + 6$,

$$\mathcal{X}_{-n} = \begin{cases} -\sum_{k=0}^{N-5}\binom{N-5}{k-1}\mathcal{X}^{(N-3k)} & \text{for } \frac{n-\ell_1}{3} \text{ even, } \frac{n-\ell_2}{3} \text{ odd} \\ \sum_{k=0}^{N-4}\left[2\binom{N-4}{k} - \binom{N-3}{k}\right]\mathcal{X}^{(N-3k)} & \text{for } \frac{n-\ell_1}{3} \text{ odd, } \frac{n-\ell_2}{3} \text{ even.} \end{cases}$$

III. Random Chebyshev billiard table knots $T(3, b)$







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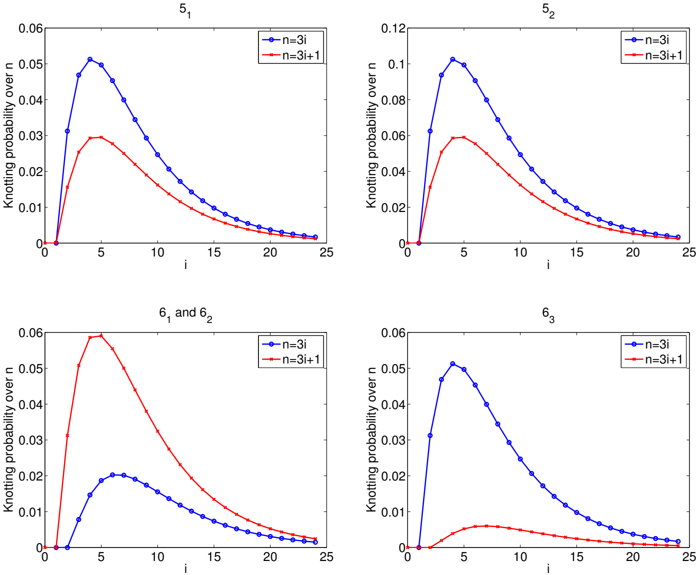


***Conjecture* [C.-Krishnan '15]:**

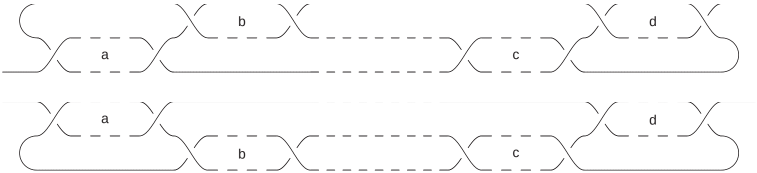
The probability of a given knot appearing in $T(3, n+1)$

is 0 for large enough n.

Question:

At what rate does this happen?

The class of ***2-bridge knots*** can be completely described using Conway notation $[a, -b, ..., (-1)^{k-1}c, (-1)^k d]$.



Because one can associate to this sequence a continued fraction, they are also called ***rational knots***.

$$\frac{\alpha}{\beta} = a + \frac{1}{-b + \frac{1}{... + \frac{1}{-c + \frac{1}{d}}}}$$

***Theorem** [Schubert '56]:*

The rational knots $\frac{\alpha}{\beta}$ and $\frac{\alpha'}{\beta'}$ are isotopic $\Leftrightarrow$

$$\alpha' = \alpha \text{ and } \beta' = \beta^{\pm 1} \bmod \alpha.$$

***'Palindrome' Theorem** [see Kauffman-Lambropoulou]:*

$\frac{\alpha}{\beta} = [a_1, \ldots, a_n]$ and $\frac{\alpha'}{\beta'} = [a_n, \ldots, a_1]$ are isotopic $\Leftrightarrow$

$$\alpha' = \alpha \text{ and } \beta\beta' \equiv (-1)^{n+1} \bmod \alpha.$$

An ***Internal Reduction Move*** Int can be performed when
$+++$ or $---$ occurs in the string.

An ***External Reduction Move*** Ext can be performed when
$++$ or $--$ occurs at the end of the string.

III. Combining reduction moves with continued fractions

Definition [Koseleff-Pecker]:

A continued fraction $[a_1, a_2, ..., a_m]$ is ***1-regular*** if $a_i \neq 0$,

(I). if $a_i a_{i+1} < 0$, then $a_{i+1} a_{i+2} > 0$ for $i = 1, \ldots, m-2$,

(E). and $a_{m-1} a_m > 0$.

Remark:

Property (I) is equivalent to *NO* internal reduction move, and

(E) is equivalent to *NO* external reduction move at the right.

Theorem [Koseleff-Pecker '11]:

$\exists!$ 1-regular continued fraction for $\frac{\alpha}{\beta}$ such that $a_i = \pm 1$.
Furthermore,

$$\frac{\alpha}{\beta} > 1 \Leftrightarrow a_2 = 1.$$

Remark:

This is equivalent to *NO* external reduction move at the right.

III. Reduced lengths for 2-bridge knots

***Definition** [modified from Koseleff-Pecker '11]:*

The ***(reduced) length*** $\ell(\frac{\alpha}{\beta})$ of the continued fraction $1 < \frac{\alpha}{\beta} = [a_1, \ldots, a_n]$ with $a_i = \pm 1$ is n.

***Proposition** [Koseleff-Pecker '11]:*

Let $\alpha > \beta > 0$ and consider $\frac{\alpha}{\beta} = PGP(\infty)$ and $N = cn(\frac{\alpha}{\beta})$.
Let β' be such that $0 < \beta' < \alpha$ and $\beta\beta' \equiv (-1)^{N-1} \mod \alpha$.
Then $\ell(\frac{\alpha}{\alpha-\beta}) + \ell(\frac{\alpha}{\beta}) = 3N - 2$ and $\ell(\frac{\alpha}{\beta'}) = \ell(\frac{\alpha}{\beta})$.

***Proposition** [Koseleff-Pecker '11]:*

Let K be a two-bridge knot with crossing number N. There exists $\frac{\alpha}{\beta} > 1$ such that K can be represented by $(\pm\frac{\alpha}{\beta})$ and $\ell(\frac{\alpha}{\beta}) < \frac{3}{2}N - 1$.

The ***mirror*** of $\frac{\alpha}{\beta} = [a_1, \ldots, a_n]$ is $-\frac{\alpha}{\beta} = [-a_1, \ldots, -a_n]$

Note that with mirrors and $a_2 \geq 1$, we have four options for β':

$$\beta, \ -\beta, \ \frac{1}{\beta}, \text{ and } \ -\frac{1}{\beta}.$$

Thus there are only four ways to write a given 2-bridge knot as $T(3, n+1)$ with no reduction moves.

Conclusion

Thanks to:

to Pierre-Vincent Koseleff as well as to

TOPOSYS

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