Arrangements of lines: when the combinatorics fails to understand the topology

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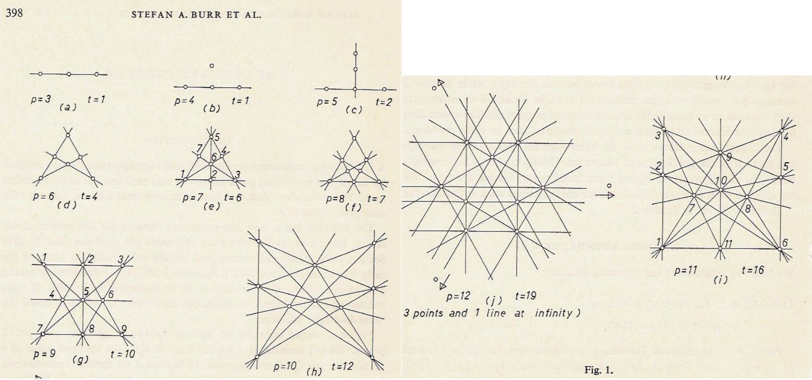
The orchard problem

(appearing in Burr-Grünbaum-Sloane 1974):



Classification of n lines with n triples

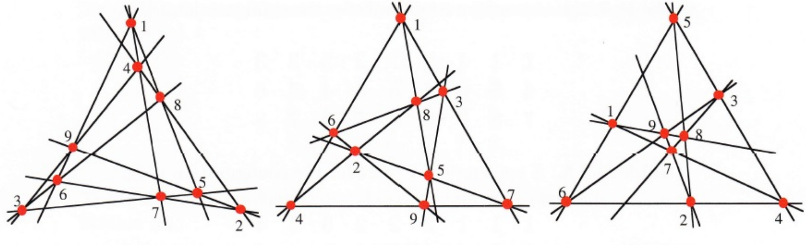
Theorem (appearing in Grünbaum 2009):

Theorem 2.2.1. *The complete list of known numbers* $\#_c(n)$, $\#_t(n)$, *and* $\#_g(n)$ *is given in Table 2.2.1.*

Table 2.2.1. The number of non-isomorphic configurations (n_3) of the three kinds, for each n. All known values are shown.

n	$\#_c(n)$	$\#_t(n)$	$\#_g(n)$
≤ 6	0	0	0
7	1	0	0
8	1	0	0
9	3	3	3
10	10	10	9
11	31	31	31
12	229	229	229
13	2,036		
14	21,399		
15	245,342		
16	3,004,881		
17	38,904,499		
18	530,452,205		
19	7,640,941,062		

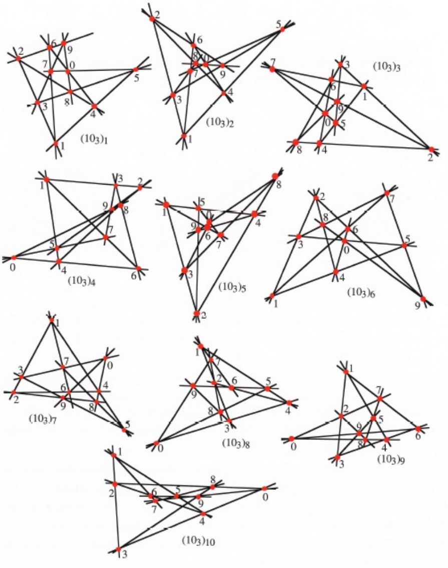
(Grünbaum attributes to Kantor 1881, Martinetti 1887, Schröter 1888, and again to Levi 1929, Hilbert and Cohn-Vossen 1932, and Gropp 1994):



$(9_3)_1$ $(9_3)_2$ $(9_3)_3$

n = 10 lines and n = 10 triples

(Grünbaum attributes to Kantor 1881 and Schröter 1889):



Classification up to eight lines via wiring diagrams

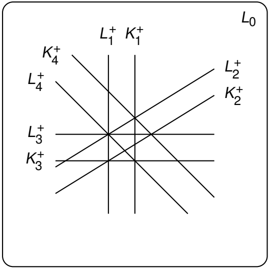
Theorem (Garber-Teicher-Vishne 2003):

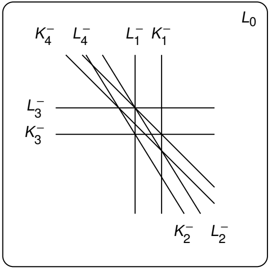
D. Garber et al. / Topology 42 (2003) 265–289

Table 1

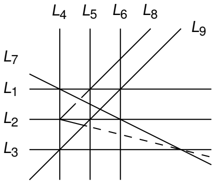
σ	τ	μ	$\triangleq$	Number of classes
−	−	−	−	3317776
−	−	−	+	306328
−	−	+	−	207361
−	−	+	+	10948
−	+	−	−	1658965
−	+	−	+	125186
−	+	+	−	103719
−	+	+	+	5543
+	−	−	−	35100
+	−	−	+	460
+	−	+	−	723
+	−	+	+	18
+	+	−	−	17644
+	+	−	+	259
+	+	+	−	723
+	+	+	+	18

The Falk-Sturmfels arrangements:





The Nazir-Yoshinaga arrangement:



Classification for 10 lines with a point of higher multiplicity

Theorem (Amram-Teicher-Ye 2013):

Let $\mathcal{A}$ be an arrangement of 10 lines with a point of multiplicity ≥ 4.
Then π is combinatorially invariant except possibly when:
$\mathcal{A}$ contains a Falk-Sturmfels arrangement or $\mathcal{A}$ is: (where $p = xy(x-z)(y-z)$)

$$p \cdot z(x-tz)(y+(1-t)z)(y-x-(1-t)z)(y-x+tz)(y-(2-t)x) = 0, \qquad t = \frac{1 \pm \sqrt{5}}{2}. \tag{1}$$

$$p \cdot z(x-t^2z)(y-(t+1)z)(y-x)(y-tx-z)(y-tx-(t^2-1)z) = 0, \qquad t^3 - t^2 + 1 = 0. \tag{2}$$

$$p \cdot z(x-t^2z)(y-tz)(y-x)(y+\frac{1}{t}x-tz)(y+\frac{1}{t}(x-z)) = 0, \qquad t = \frac{-1 \pm \sqrt{5}}{2}. \tag{3}$$

$$p \cdot z(x-(1 \pm \sqrt{2}/2)z)(y-(2 \pm \sqrt{2})z)(y-x)(y \mp \sqrt{2}x \pm \sqrt{2}z)(y \mp \sqrt{2}x - z) = 0. \tag{4}$$

$$p \cdot (x-y)(x-\frac{t-1}{t}z)(y-tz)(y-\frac{t}{t+1}z)(y-\frac{t^2+t+1}{t+1}x-\frac{t}{t+1}z)(y-\frac{t^2-t+1}{t^2}(x+\frac{1}{t}z)) = 0, \qquad t^3-t^2+1=0. \tag{5}$$

$$p(x-tz)(y-\frac{t+1}{t}z)(y-(t^2+t)z)(y-(t+1)x)(y+\frac{t+1}{t}x-\frac{t+1}{t}z)(y-\frac{1}{1-t}(x-tz)) = 0, \qquad t^3-t^2-2t+1=0. \tag{6}$$

$$p \cdot (x-tz)(y-tz)(y+\frac{t}{(t-1)^2}z)(y-x)(y-\frac{t}{1-t}(x-tz))(y-\frac{t}{(1-t)^2}(x-z)) = 0, \qquad t^3-2t^2+3t-1=0. \tag{7}$$

$$p \cdot (x-y)(x-tz)(y-tz)(y-\frac{t^2}{t-1}z)(y-\frac{1}{t-1}(x-z))(y+\frac{t^2}{t(t-1)}x-\frac{t^2}{t-1}z) = 0, \qquad t^3-2t^2+t-1=0. \tag{8}$$

$$p(x-tz)(y-tz)((t-1)y-t^2z)(x-y)(tx+(t-1)y-t^2z)((t^2-t+1)x-(t-1)y+(t-1)z) = 0, t^4-2t^3+4t^2-3t+1=0. \tag{9}$$

Classification for 10 lines with only triples and doubles

Theorem (Amram-C-Teicher-Ye):

Case	#comb	#non-geom	#geom	$\#\mathcal{Z}$	#irred/$\mathbb{C}$
10.	10	1	9		9
13.	2	2			
(9_3).i.	5		5		5
(9_3).ii.	4		4	1+1	2
(9_3).iii.	5	2	3	2	1
12.B.3.	4	1	3	2+1	
12.B.2.	4	2	2	1	1
12. total	22	5	17	8	9
11.A.	10		10		10
11.B.2.	4		4		4
11.B.3.a.	3		3		3
11.B.3.b.2.	7		7	1	6
11.B.3.b.1.	13	1	12		12
11. total	37	1	36	1	35
Total	71	9	62	5+4	53

Theorem (Amram-C-Teicher-Ye):

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_8	e_8	e_8	e_4	e_3	e_2	D
e_2	e_4	e_6	e_4	e_2	e_3	e_7	e_5	e_5	F
e_3	e_5	e_7	e_6	e_7	e_9	e_9	e_6	e_9	I
I	D	F	I	D	F				

The arrangement (9_3).ii.DFI.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_8	e_8	e_8	e_4	e_3	e_2	C
e_2	e_4	e_6	e_4	e_2	e_3	e_7	e_5	e_5	F
e_3	e_5	e_7	e_6	e_7	e_9	e_9	e_6	e_9	I
I		F	I	C	F		C		

The arrangement (9_3).ii.CFI.

e_1	e_1	e_1	e_8	e_8	e_8	e_9	e_9	e_9	B
e_2	e_4	e_6	e_2	e_5	e_3	e_2	e_4	e_3	D
e_3	e_5	e_7	e_4	e_6	e_7	e_5	e_7	e_6	F
	F	B	B		D	D		F	

The arrangement (9_3).iii.BDF.

e_1	e_1	e_1	e_8	e_8	e_8	e_9	e_9	e_9	A
e_2	e_4	e_6	e_2	e_5	e_3	e_2	e_4	e_3	C
e_3	e_5	e_7	e_4	e_6	e_7	e_5	e_7	e_6	G
G		C	A	G		C		A	

The arrangement (9_3).iii.ACG.

L_1	L_2	L_3	L_{10}	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_6	e_6	e_{11}	e_2	e_{11}	e_{11}	e_{12}	e_{12}
e_2	e_4	e_7	e_9	e_{12}	e_4	e_3	e_8	e_3	e_5
e_3	e_5	e_8	e_{10}	e_1	e_7	e_5	e_{10}	e_{10}	e_8
				e_7	e_9	e_6	e_2	e_4	e_9

The arrangement 12.B.2.iv.

L_1	L_2	L_3	L_{10}	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_4	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_5	e_4
e_3	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_7	e_6	e_6
				e_9	e_5	e_7	e_8	e_8	e_9

The arrangement 12.B.3.a.i.

e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_3	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_7	e_7
e_4	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_5	e_9	e_8
				e_8	e_9	e_6	e_6	e_4	e_5

The arrangement 12.B.3.b.ii.

e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_3	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_7	e_7
e_4	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_5	e_9	e_8
				e_8	e_9	e_6	e_6	e_5	e_4

The arrangement 12.B.3.b.iii.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_{10}	e_6	e_4	e_7	e_8	e_6	e_9
e_4	e_6	e_8	e_{11}	e_{10}	e_{11}	e_8	e_{10}	e_9	e_{11}
e_5	e_7	e_9							

The arrangement 11.B.3.b.2.v.

TABLE 7. The nine potential Zariski pairs that arise from this present classification.

Theorem (Amram-C-Sun-Teicher-Ye-Zarkh):

For arrangements {1}, {6}, and {7} with $\mathbb{Z}_2 \subseteq \mathrm{Aut}(\mathcal{A})$, $\varphi : x \mapsto y$, $y \mapsto x$, and $z \mapsto z$ is a homeo. btwn the complements.

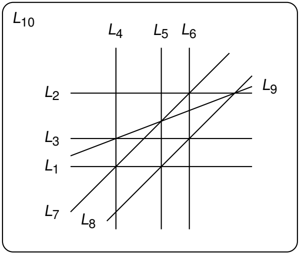
Case by section	Over	Aut($\mathcal{A}$)	Algorithm?
{1}: Eqn (1)	$\mathbb{R}$	$\mathbb{Z}_2$	Y
{6}: (9_3).iii.ACG.	$\mathbb{R}$	$\mathbb{Z}_2$	Y
{7}: (9_3).iii.BDF.	$\mathbb{R}$	S_4	Y
MacLane	$\mathbb{C}$	$GL(2;\mathbb{F}_3)$	Y
Nazir-Yoshinaga	$\mathbb{C}$	S_3	Y
11.B.3.b.2.iii.	$\mathbb{C}$	$\mathbb{Z}_2$	Y
11.B.3.b.2.iv.	$\mathbb{C}$	$\mathbb{Z}_2$	Y
11.B.2.iv.	$\mathbb{C}$	$\mathbb{Z}_2$	Y
Falk-Sturmfels	$\mathbb{R}$	$\mathbb{Z}_4$	N
Rybnikov	$\mathbb{C}$	$S_3 \times \mathbb{Z}_2$	N

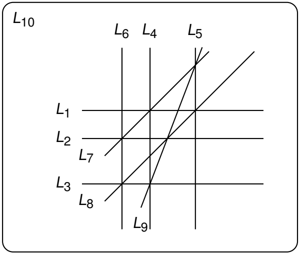
From Equation (1) in [ATY] we obtain the defining equation

$$0 = xyz(x-z)(y-z)(x-tz)(y+(1-t)z)$$

$$(y-x-(1-t)z)(y-x+tz)(y-(2-t)x),$$

where $t^{\pm} = \frac{1\pm\sqrt{5}}{2}$.





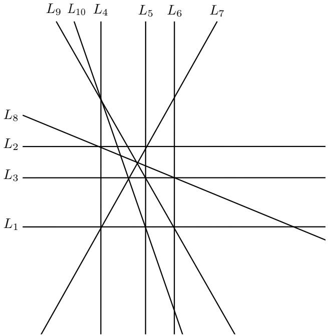
The two arrangements in the moduli space for {6}

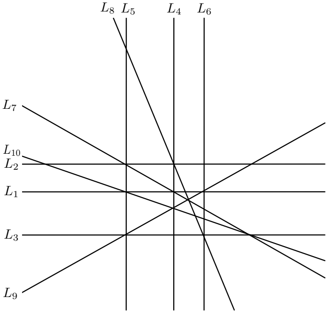
From (9_3).iii.ACG. in [ACTY] we obtain the defining equation

$$0 = xy(x - z)(y - z)(x - tz)(y + t^{-1}z)$$

$$(x+y-z)(ty-(1+t^{-1})x+z)((t-1)y-t^{-1}x+z)((t-1)y-(1+t^{-1})x+z),$$

where $t^{\pm} = \frac{-1 \pm \sqrt{5}}{2}$.





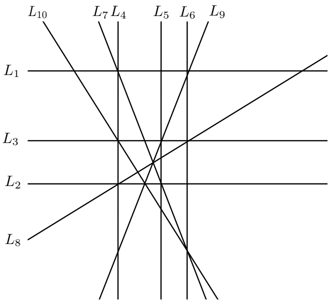
The two arrangements in the moduli space for {7}

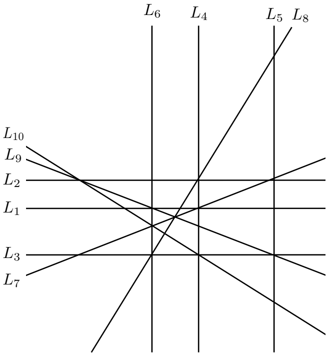
From (9_3).iii.BDF. in [ACTY] we obtain the defining equation

$$0 = xy(x-z)(y-z)(x-tz)(y+t^{-1}z)$$

$$(x+y)(tx+y-z)(tx-y-tz)(x-t^2y-tz),$$

where $t^{\pm} = \frac{1\pm\sqrt{5}}{2}$.





Meirav Amram, Mina Teicher, and Fei Ye, *Moduli spaces of arrangements of 10 projective lines with quadruple points*, Adv. in Appl. Math. **51** (2013), no. 3, 392–418.

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Meirav Amram, Moshe Cohen, Hao Sun, and Mina Teicher, *A distance between real arrangements and examples*, (in preparation).