A dimer model for the Jones polynomial of pretzel knots

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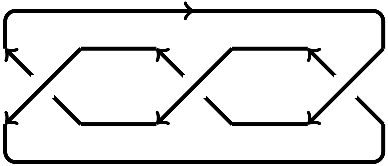
A ***knot*** K is S^1 embedded in S^3. We ***orient*** the knot.

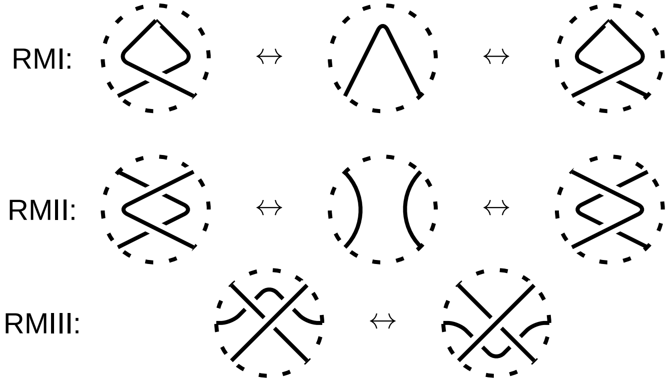
A ***knot diagram*** D is the projection of the knot onto $\mathbb{R}^2$ with under- and over-crossing information.



Theorem (Reidemeister 1926):

Two diagrams represent the same knot $\Leftrightarrow$

$\exists$ a sequence of Reidemeister moves taking one to the other.



A ***knot invariant*** is an evaluation on a knot diagram that is constant under each of the three *Reidemeister moves*.

Walk-through

Motivation:

- Jones polynomial of K $\leftrightarrow$ Tutte polynomial of Tait graph G.
- Activity gives spanning tree model for Tutte polynomial.
- Champanerkar-Kofman: spanning tree model for $\widetilde{Kh}$.
- Kronheimer-Mrowka: $\widetilde{Kh}$ detects the unknot.

Goals:

- Spanning trees of $G \longleftrightarrow$ perfect matchings of Γ.
- List of perfect matchings as a matrix determinant.
- Jaeger-Vertigan-Welsh: Jones polynomial is $\#P$-hard.
- ...but pretzel knots work!

$$
\left(\begin{array}{ccc|ccc|c|ccc|c|cccc}
L & & & & & & & & & & & \ell & & & \\
D & \ddots & & & & & & & & & & d & & & \\
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 & & & L & & & & & & & D & d & \ell & & \\
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 & & & & \ddots & L & & & & & & \vdots & \vdots & & \\
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 & & & & & & & L & & & D & & & & d \\
 & & & & & & & D & \ddots & & & & & & d \\
 & & & & & & & & \ddots & L & & & & & \vdots \\
 & & & & & & & & & D & & & & & d
\end{array}\right)
$$

Graphs from knots: the signed Tait graph G

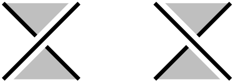
A ***signed graph*** has edges weighted $+1$ or -1.

Checkerboard color the regions of a knot diagram D.

Definition:

The ***signed Tait graph*** G associated with D has

$V(G) = \{\text{colored regions}\}$ and $E(G) = \{\text{crossings of } D\}$.

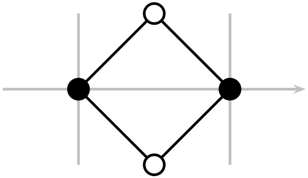
positive  negative

Note that the dual G^* comes from the uncolored regions.

Graphs from knots: the overlaid Tait graph $\widehat{\Gamma}$

Definition:

The ***overlaid Tait graph*** $\widehat{\Gamma}$ associated with D is bipartite with $V(\widehat{\Gamma}) = [E(G) \cap E(G^*)] \sqcup [V(G) \sqcup V(G^*)]$ and $E(\widehat{\Gamma})$ the half-edges of G and G^*.

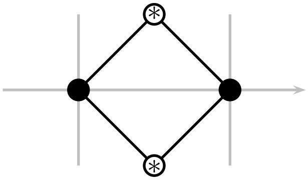


Each face in the overlaid Tait graph $\widehat{\Gamma}$ is a square.

Graphs from knots: the balanced overlaid Tait graph Γ

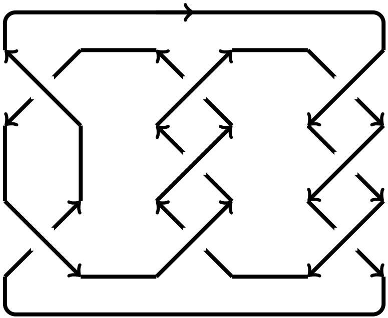
Definition:

The ***balanced overlaid Tait graph*** Γ associated with D is obtained from $\widehat{\Gamma}$ by removing two vertices from the larger set that lie on the same face:



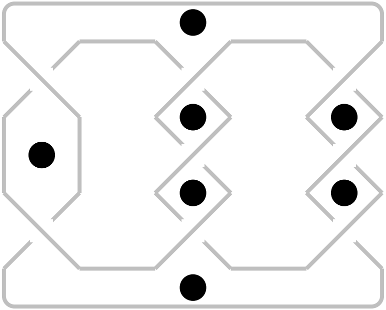
"Balanced" means the two vertex sets are the same size.

Graphs from knots: the signed Tait graph G



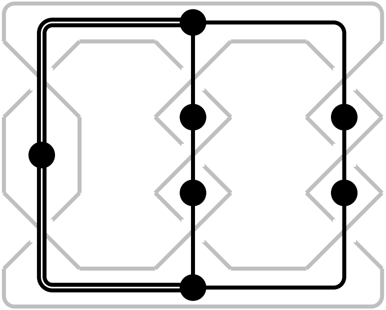
The oriented knot 8_{19},

Graphs from knots: the signed Tait graph G



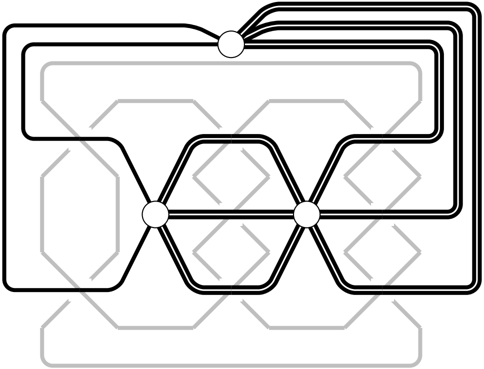
a checkerboard coloring,

Graphs from knots: the signed Tait graph G



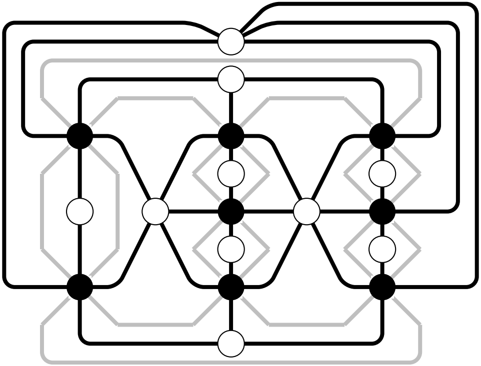
the corresponding signed Tait graph G,

Graphs from knots: the signed Tait graph G



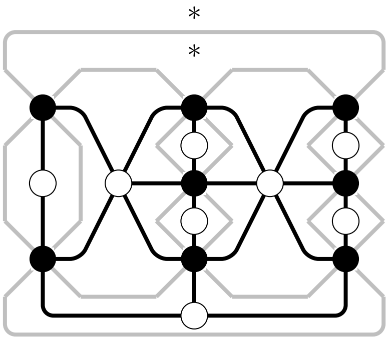
the dual signed Tait graph G^*,

Graphs from knots: the balanced overlaid Tait graph Γ



the overlaid Tait graph $\widehat{\Gamma}$ (all faces are square),

Graphs from knots: the balanced overlaid Tait graph Γ



and the balanced overlaid Tait graph Γ.

Tutte's activity words: Definition

Definition (Tutte's Activity words):

For spanning tree S of signed graph G with ordered edges, assign an activity letter to each edge:

$+$	live	dead	$-$	live	dead
internal	L	D	internal	$\overline{L}$	$\overline{D}$
external	ℓ	d	external	$\overline{\ell}$	$\overline{d}$

Activity ("live" or "dead") is determined by the ordering:

Tutte's activity words: Definition

For external edge $e \notin S$, there is a unique cycle in $S \cup \{e\}$.

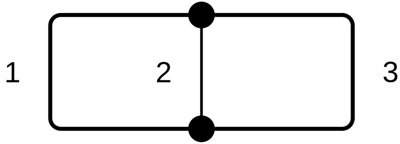
$e \notin S$ is live if it is the lowest-ordered edge in the cycle.

For internal edge $e \in S$, the graph $S \backslash \{e\}$ is disconnected.

$e \in S$ is live if it is the lowest-ordered edge that reconnects.

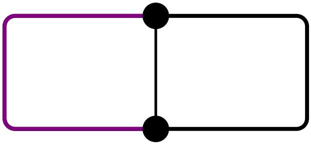
Let $a(e, S)$ be the ***activity letter*** for the edge e and the tree S, and let $a(S)$ be the ***activity word*** associated to the tree S.

Tutte's activity words: Example



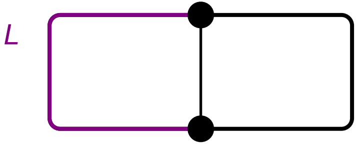
For the (all positive) graph G

Tutte's activity words: Example



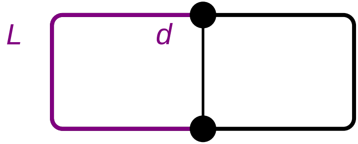
and the spanning tree S_1,

Tutte's activity words: Example



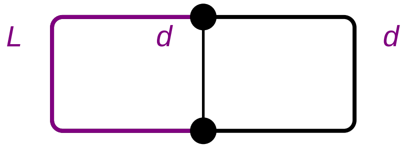
the first edge is L,

Tutte's activity words: Example



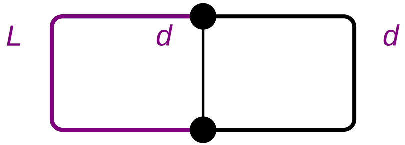
the second edge is d,

Tutte's activity words: Example



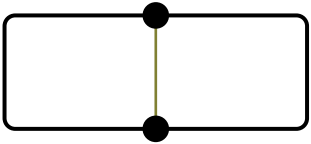
and the third edge is also d,

Tutte's activity words: Example



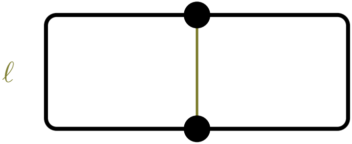
giving the activity word $a(S_1) = (Ldd)$.

Tutte's activity words: Example



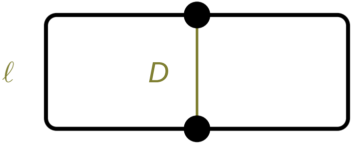
For the spanning tree S_2,

Tutte's activity words: Example



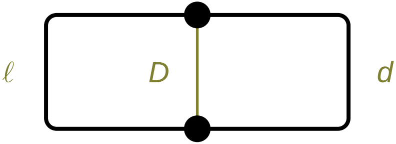
the first edge is ℓ,

Tutte's activity words: Example



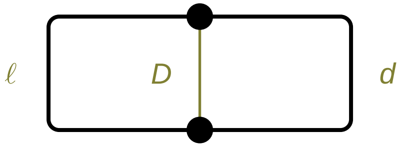
the second edge is D,

Tutte's activity words: Example



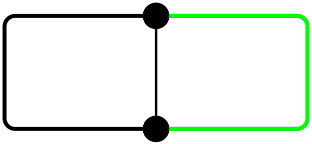
and the third edge is d,

Tutte's activity words: Example



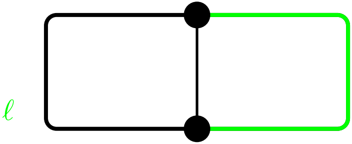
giving the activity word $a(S_2) = (\ell D d)$.

Tutte's activity words: Example



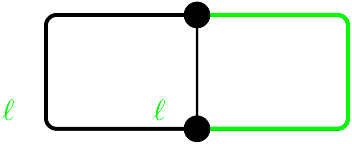
And for the spanning tree S_3,

Tutte's activity words: Example



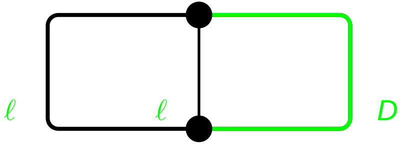
the first edge is ℓ,

Tutte's activity words: Example



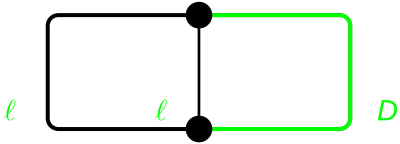
the second edge is ℓ,

Tutte's activity words: Example



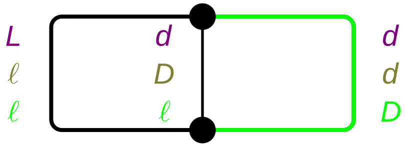
and the third edge is *D*,

Tutte's activity words: Example



giving the activity word $a(S_3) = (\ell\ell D)$.

Tutte's activity words: Example



Thus the activity words are (Ldd), (ℓDd), and $(\ell\ell D)$.

Tutte polynomial $T(G; x, y)$

For (unsigned) graph G and edge e,

let $G \backslash e$ be the deletion of e and G/e the contraction.

Definition (Tutte):

The (unsigned) *Tutte polynomial* $T(G; x, y) =$

$$\begin{cases} T(G \backslash e; x, y) + T(G/e; x, y) & \text{if } e \text{ is neither a bridge nor a loop,} \\ x^{\#\text{ bridges}} y^{\#\text{ loops}} & \text{if all edges are bridges and loops.} \end{cases}$$

Theorem (Tutte):

$$T(G; x, y) = \sum_S x^{\#L} y^{\#\ell} = \sum_S \prod_{e \in E(G)} a(e, S)|_T$$

Tutte polynomial $T(G; x, y)$

$a(e,S)$	L	D	ℓ	d	$\overline{L}$	$\overline{D}$	$\overline{\ell}$	$\overline{d}$
$a(e,S)\|_T$	x	1	y	1	$--$	$--$	$--$	$--$

The activity evaluations for the Tutte polynomial $T(G; x, y)$

Signed Tutte polynomial $Q(G; A, B, \delta)$

Definition (Kauffman):

The *signed Tutte polynomial* $Q(G; A, B, \delta) =$

$$\begin{cases} AQ(G\backslash\overline{e}; A, B, \delta) + BQ(G/\overline{e}; A, B, \delta) & \text{non-bridge/loop } \overline{e}, \\ BQ(G\backslash e; A, B, \delta) + AQ(G/e; A, B, \delta) & \text{non-bridge/loop } e, \\ x^{\#\text{ bridges } + \#\ \overline{\text{loops}}}\, y^{\#\text{ loops } + \#\ \overline{\text{bridges}}} & \text{all bridges/loops,} \end{cases}$$

setting $x = A + B\delta$ and $y = A\delta + B$.

Theorem (Kauffman):

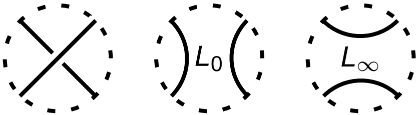
$$Q(G; A, B, \delta) = \sum_{S} \prod_{e \in E(G)} a(e, S)|_{Q}$$

Signed Tutte polynomial $Q(G; A, B, \delta)$

$a(e,S)$	L	D	ℓ	d	$\overline{L}$	$\overline{D}$	$\overline{\ell}$	$\overline{d}$
$a(e,S)\vert_Q$	x	A	y	B	y	B	x	A

The activity evaluations for the signed Tutte polynomial $Q(G; A, B, \delta)$ with $x = A + B\delta$ and $y = A\delta + B$

Kauffman bracket polynomial $\langle K \rangle$



Definition (Kauffman):

The *Kauffman bracket polynomial* $\langle L \rangle$ of link L satisfies

1. Smoothing relation: $\langle L \rangle = A\langle L_0 \rangle + A^{-1}\langle L_\infty \rangle$
2. Stabilization: $\langle U \sqcup L \rangle = (-A^2 - A^{-2})\langle L \rangle$
3. Normalization: $\langle U \rangle = 1$.

For knot K with signed Tait graph G,

Theorem (Thistlethwaite):

$$\langle K \rangle = \sum_S \prod_{e \in E(G)} a(e,S)|_V$$

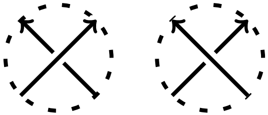
Kauffman bracket polynomial $\langle K \rangle$

$a(e,S)$	L	D	ℓ	d	$\overline{L}$	$\overline{D}$	$\overline{\ell}$	$\overline{d}$
$a(e,S)\vert_V$	$-A^{-3}$	A	$-A^{3}$	A^{-1}	$-A^{3}$	A^{-1}	$-A^{-3}$	A

The activity evaluations for the Kauffman bracket $\langle K \rangle$

Jones polynomial $V_K(t)$

The ***writhe*** $w(D)$ of an oriented diagram is the sum:

$+1$  -1

Definition (Jones):
The *Jones polynomial* $V_L(t)$ of link L satisfies, for $A = t^{-1/4}$,

$$V_L(t) = (-A^{-3})^{w(D)} \langle L \rangle .$$

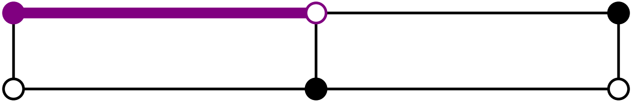
For a knot K with signed Tait graph G,

Theorem (Thistlethwaite):

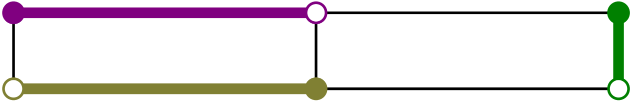
$$V_K(t) = (-A^{-3})^{w(D)} \sum_S \prod_{e \in E(G)} a(e,S)|_V$$

Dimer model

A ***dimer*** in a (bipartite) graph is just an edge.



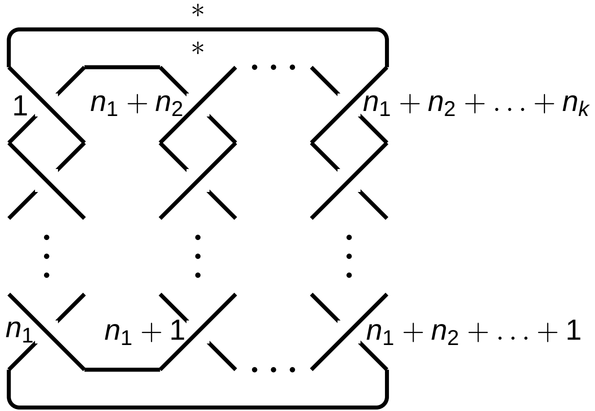
A ***perfect matching*** μ is a collection of non-incident dimers that covers the graph.



The correspondence between G and Γ

signed Tait graph G	balanced overlaid Tait graph Γ
edge $e \in E(G)$	edge $\varepsilon \in E(\Gamma)$
squared incidence matrix	bipartite adjacency submatrix
rooted spanning tree S in G	perfect matching μ in Γ
activity $a(e, S)$	activity weighting $\alpha(\varepsilon)$

$P = P(n_1, n_2, \ldots, n_k)$-pretzel knot



The $(n_1, n_2, \ldots, n_k)$-pretzel knot P
with an ordering on the crossings.

Main results: activity words

Main Theorem:

Summing over all perfect matchings μ in Γ

and taking the product over all edges $\varepsilon \in \mu$,

$$\sum_{\mu} \prod_{\varepsilon \in \mu} \alpha(\varepsilon) = \sum_{S} a(S)$$

gives the complete list of activity words $a(S)$ associated with spanning trees S of G associated with the diagram of P.

Main results: Jones polynomial

Main Corollary:

Summing over all perfect matchings μ in Γ
and taking the product over all edges $\varepsilon \in \mu$,

$$\sum_{\mu} \prod_{\varepsilon \in \mu} w(\varepsilon)\alpha(\varepsilon)|_V = V_P(t)$$

gives the Jones polynomial $V_P(t)$ of P.

Main results: matrix determinant

Computational Corollary:

Let ε_{ij} be the edge $\varepsilon \in E(\Gamma)$ btwn the i-th vertex coming from the crossings and the j-th vertex coming from the regions.

Let $A = (\kappa(\varepsilon_{ij})w(\varepsilon_{ij})\alpha(\varepsilon_{ij})|_V)$ be the activity weighting on the bipartite adjacency submatrix associated with P. Then

$$\det(A) = V_P(t)$$

gives the Jones polynomial $V_P(t)$ of P up to sign.

A note on pretzel knots

The results above hold for pretzel knots $\forall k \in \mathbb{N}$, $|n_i| \in \mathbb{N}$.

One cannot hope to achieve this result for a general knot K.

Theorem (Jaeger-Vertigan-Welsh):

Determining the Jones polynomial is $\#P$-hard.

Matrices from graphs: the incidence matrix

The ***incidence matrix*** has rows labelled by edges and columns labelled by vertices.

$m_{ij} = 0$ if the i-th edge is not incident with the j-th vertex.

This $|E| \times |V|$ matrix is in general not square.

The ***squared incidence matrix*** is the incidence matrix of the graph together with the incidence matrix for the dual graph with a column of each deleted.

This $|E| \times [(|V| - 1) + (|F| - 1)]$ matrix is square.

Matrices from graphs: the adjacency matrix

The ***adjacency matrix*** rows and columns labelled by vertices.

$m_{ij} = 0$ if the i-th vertex is not adjacent to the j-th vertex.

For a bipartite graph, present this square matrix in block form

$$\left(\begin{array}{c|c} 0 & M \\ \hline M^T & 0 \end{array}\right)$$

The ***bipartite adjacency submatrix*** is the block M.

Proposition:

The squared incidence matrix of the Tait graph G is the bipartite adjacency submatrix of the balanced overlaid Tait graph Γ.

Matrices from graphs: determinant and permanent

Recall the determinant of a matrix $M = (m_{ij})$

$$det(M) = \sum_{\sigma \in \mathfrak{S}} \prod_i (-1)^{sign(\sigma)} m_{i\sigma(i)}$$

The ***permanent*** or ***unsigned determinant*** is

$$perm(M) = \sum_{\sigma \in \mathfrak{S}} \prod_i m_{i\sigma(i)}$$

Matrices from graphs: determinant and permanent

Proposition:

The terms in the permanent expansion of a bipartite adjacency submatrix associated with a(n unsigned) balanced bipartite graph give the complete list of perfect matchings of the graph.

Proof:

Each term in the permanent expansion is a permutation σ matching each vertex i in the first vertex set to a vertex $\sigma(i)$ in the second vertex set. $\square$

Kauffman's trick $\kappa(\varepsilon)$: signing the entries

This will be used to sign the corresponding entries in the matrix.

A ***Kasteleyn weighting*** of a plane bipartite graph is a signing of the edges such that $\#$ negatives around a particular face is

- odd if the face has length 0 mod 4 or
- even if the face has length 2 mod 4.

Lemma:

Suppose G has a Kasteleyn weighting. Then so does $G \backslash e$.

Kauffman's trick $\kappa(\varepsilon)$: signing the entries

Proof:

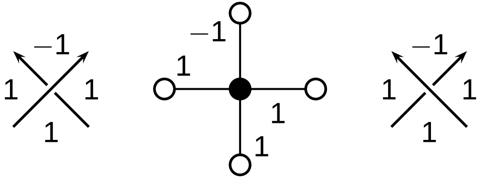
Let e be incident with with two faces of length f_1 and f_2.
Delete e to replace these with a face of length $f_1 + f_2 - 2$.

f_1	$\#$ negs	f_2	$\#$ negs	$f_1 + f_2 - 2 \bmod 4$	$\#$ negs
0	odd	0	odd	2	even
0	odd	2	even	0	odd
2	even	0	odd	0	odd
2	even	2	even	2	even

Then $\#$ negs changes by 0 or 2 (an even number) compared with the sum of $\#$ negs in f_1 and f_2. $\square$

Kauffman's trick $\kappa(\varepsilon)$: signing the entries

Kauffman's trick $\kappa(\varepsilon)$ to distribute signs to the edges of the balanced overlaid Tait graph Γ coming from a knot diagram:



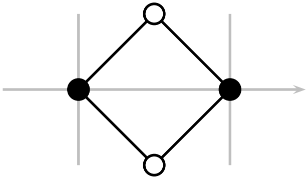
Proposition:

Kauffman's trick $\kappa(\varepsilon)$ provides a Kasteleyn weighting.

Kauffman's trick $\kappa(\varepsilon)$: signing the entries

Proof:

Each face in the overlaid Tait graph $\widehat{\Gamma}$ is a square. The balanced overlaid Tait graph Γ is obtained by edge deletions.



The assigning of a negative edge affects exactly one of the NW and SW sides of the square. □

Kauffman's trick $\kappa(\varepsilon)$: signing the entries

Proposition:

The determinant expansion of a bipartite adjacency submatrix associated with a Kasteleyn-weighted balanced bipartite graph gives the complete list of perfect matchings up to sign.

Proof:

Two permutations differ by a transposition $\longleftrightarrow$
$\exists$ four non-zero terms in a rectangle in the matrix $\longleftrightarrow$
$\exists$ a square face in the graph.
$\exists!$ negative sign in each square, so these have
opposite signs in both the matrix and the perfect matching. $\square$

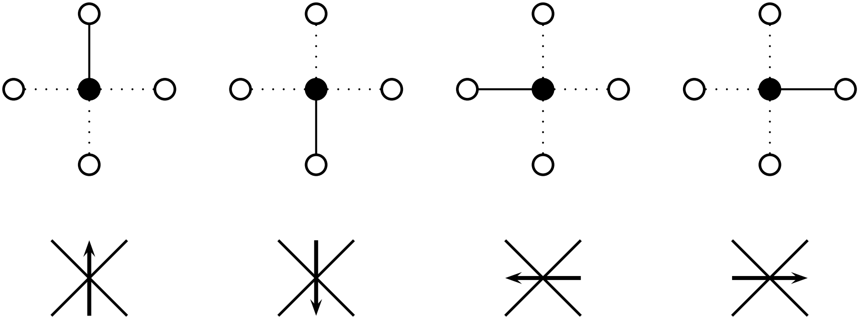
Proposition:

Given a knot diagram, there is a bijection between perfect matchings of the balanced overlaid Tait graph Γ and rooted spanning trees of the Tait graph G.

Proof:

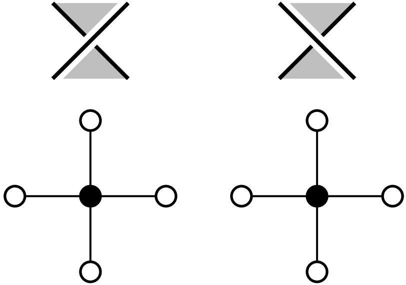
$\{$perfect matchings of $\Gamma\} \cong$
$\{$permanent expansion of the bipartite adjacency submatrix$\} \cong$
$\{$permanent expansion of the squared incidence matrix$\} \cong$
$\{$partition of edges $T \subset G$ and $T^c \subset G^*\}$
T spans; if $\exists$ cycle C, then $*$ must be on one side of C.
T^c spans; $\exists$ cycle in the dual on the same side of C.
Repeat this process, yielding an infinite graph. $\rightarrow\leftarrow$ $\square$

Correspondence between edges ε of the overlaid Tait graph $\widehat{\Gamma}$ and directed edges e of the (directed) Tait graph G.



Writhe weighting $w(\varepsilon)$: edges $\varepsilon \in E(\Gamma)$

The ***writhe weighting*** $w(\varepsilon)$ on $\varepsilon \in E(\Gamma)$ is $(-A)^{-3}$ or $(-A)^3$:



$(-A)^{-3}$ $(-A)^3$

Writhe weighting $w(\varepsilon)$: bipartite adjacency submatrix

Let ε_{ij} be the edge $\varepsilon \in E(\Gamma)$ btwn the i-th vertex coming from the crossings and the j-th vertex coming from the regions.

The ***writhe weighting*** $w(\varepsilon_{ij})$ is determined by the sign of the i-th vertex coming from the crossings.

At the level of the bipartite adjacency submatrix, this means multiplying all entries in each row by $(-A)^{-3}$ or $(-A)^{3}$.

Activity weighting $\alpha(\varepsilon)$: edges $\varepsilon \in E(\Gamma)$

The bipartition of the vertices in Γ is really the tripartition

$$V(\Gamma) = [E(G) \cap E(G^*)] \sqcup [V(G)] \sqcup [V(G^*)] = V_E \sqcup V_V \sqcup V_F$$

Definition

The ***activity weighting*** $\alpha(\varepsilon)$ on $\varepsilon = v_i v_j \in E(\Gamma)$ is given by:

an edge incident with $v_i \in V_E$ is $+$ or $-$ if $e \in E(G)$ is $+$ or $-$;

an edge incident with $v_j \in V_V$ is internal, and
an edge incident with $v_j \in V_F$ is external; and

an edge is live if it connects the lowest-ordered $v_i \in V_E$
to the vertex $v_j \in V_V \sqcup V_F$ and dead otherwise.

Activity weighting $\alpha(\varepsilon)$: bipartite adjacency submatrix

The entries of the bipartite adjacency submatrix associated to the balanced overlaid Tait graph Γ obey the following rules:

ordered rows associated with V_E are all positive or all negative;

columns associated with V_V are internal and V_F are external;

the first non-zero entry in a column is live, the rest are dead.

$$\left(\begin{array}{ccc|ccc|c|ccc|c|cccc}
L & & & & & & & & & & & \ell & & & \\
D & \ddots & & & & & & & & & & d & & & \\
& \ddots & L & & & & & & & & & \vdots & & & \\
& & D & & & & & & & & L & d & & & \\
\hline
& & & L & & & & & & & D & d & \ell & & \\
& & & D & \ddots & & & & & & & d & d & & \\
& & & & \ddots & L & & & & & & \vdots & \vdots & & \\
& & & & & D & & & & & & d & d & & \\
\hline
& & & & & & \ddots & & & & \vdots & & \vdots & \ddots & \vdots \\
\hline
& & & & & & & L & & & D & & & & d \\
& & & & & & & D & \ddots & & & & & & d \\
& & & & & & & & \ddots & L & & & & & \vdots \\
& & & & & & & & & D & & & & & d
\end{array}\right)$$

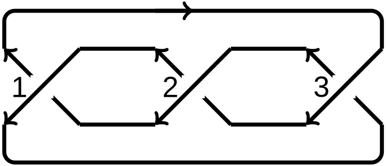
A note on the proof

The proof that the terms in the determinant expansion give the ***exact activity words*** for the pretzel knots comes from a technical lemma (C.) on the activity of paths.

One difficulty to extending this class is producing a complete list of activity words for more general knots.

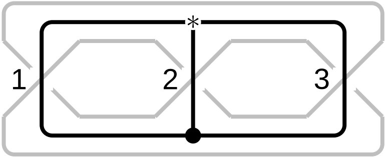
Example 1: the Jones polynomial for the trefoil

Example 1: the $(1, 1, 1)$-pretzel knot



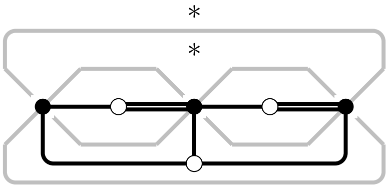
Example 1: the Jones polynomial for the trefoil

Tait graph G



Example 1: the Jones polynomial for the trefoil

balanced overlaid Tait graph Γ



Example 1: the Jones polynomial for the trefoil

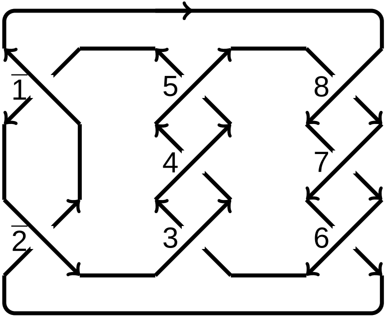
The spanning trees give activity words (Ldd), (ℓDd), and $(\ell\ell D)$:

$$\left(\begin{array}{c|cc} L & \ell & \\ \hline D & -d & \ell \\ \hline D & & -d \end{array}\right)$$

With writhe $(-A^{-3})^{-3}$, the determinant is $A^4 + A^{12} - A^{16} = t^{-1} + t^{-3} - t^{-4}$, the Jones polynomial of the trefoil.

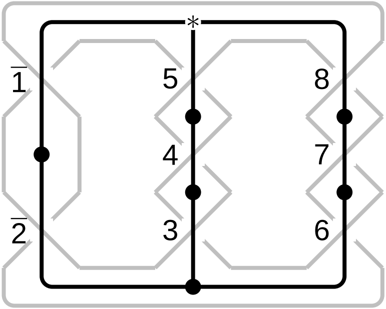
Example 2: the Jones polynomial for 8_{19}

Example 2: the $(-2, 3, 3)$-pretzel knot



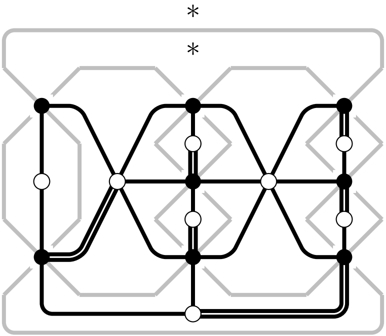
Example 2: the Jones polynomial for 8_{19}

Tait graph G



Example 2: the Jones polynomial for 8_{19}

balanced overlaid Tait graph Γ



Example 2: the Jones polynomial for 8_{19}

$$\left(\begin{array}{c|cc|cc|c|cc}
\overline{L} & & & & & & \overline{\ell} & \\
\overline{D} & & & & & \overline{L} & -\overline{d} & \\
\hline
 & -L & & & & D & d & \ell \\
 & D & -L & & & & d & d \\
 & & D & & & & d & d \\
\hline
 & & & L & & -D & & d \\
 & & & -D & L & & & d \\
 & & & & -D & & & d
\end{array}\right)$$

With writhe $(-A^{-3})^8$, the determinant is $-A^{-32} + A^{-20} + A^{-12}$ $= -t^8 + t^5 + t^3$, the Jones polynomial of 8_{19}.

Example 3: the Jones polynomial for the $(-2,3,7)$-pretzel knot

$$\left(\begin{array}{c|cc|cccccc|c|cc}
\overline{L} & & & & & & & & & & \overline{\ell} & \\
\overline{D} & & & & & & & & & \overline{L} & -\overline{d} & \\
\hline
 & -L & & & & & & & & D & d & \ell \\
 & D & -L & & & & & & & & d & d \\
 & & D & & & & & & & & d & d \\
\hline
 & & & L & & & & & & -D & & d \\
 & & & -D & L & & & & & & & d \\
 & & & & -D & L & & & & & & d \\
 & & & & & -D & L & & & & & d \\
 & & & & & & -D & L & & & & d \\
 & & & & & & & -D & L & & & d \\
 & & & & & & & & -D & & & d
\end{array}\right)$$

With writhe $(-A^{-3})^{12}$, the determinant is
$-A^{-40}+A^{-36}-A^{-32}+A^{-16}+A^{-8}=-t^{10}+t^{9}-t^{8}+t^{2}$.

Leaving the class of pretzel knots

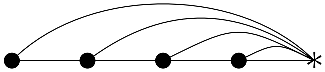
Property: (Subdivision/Doubling)

Let $e_n \in E(G)$ be incident with the omitted vertex and face.

Then if the activity weighting on Γ provides a dimer model for G,

this can be extended to one for $G \cup \{e_{n+1}\}$

that subdivides or doubles e_n.



Leaving the class of pretzel knots

Proof:

Row e_n in squared incidence matrix only has D and d.

Subdivide to get a new row e_{n+1} and a new vertex column.

Entries in this column are 0 except for L and D.

Determinant expansion terms give DD and dD, preserving the first n pivots, or Ld, preserving the first $n-1$ pivots.

These cases are exactly the possibilities for activity words.

The dual case of doubling works similarly. $\square$

Another corollary to the Main Theorem

Reduced Khovanov homology chain complex $\widetilde{CKh}$**:**

$a(e,S)$	L	D	ℓ	d	$\bar{L}$	$\bar{D}$	$\bar{\ell}$	$\bar{d}$
$a(e,S)\|_K$	uv	v	u^{-1}	1	u^{-1}	1	u	1

Another corollary to the Main Theorem

Corollary:

Summing over all perfect matchings μ in Γ
and taking the product over all edges $\varepsilon \in \mu$,

$$\sum_{\mu} \prod_{\varepsilon \in \mu} \alpha(\varepsilon)|_K$$

gives the two-variable polynomial $\widetilde{CKh}_P(q, t)$ for
the reduced Khovanov chain complex of P up to sign.

Questions

What can these easy computations teach us about the Jones polynomial of the class of pretzel knots?

The activity weighting can be extended to a larger class of knots, but how far can it go?

The first-order differential of reduced Khovanov homology can be found in the activity matrix, but the higher-order ones?