Kauffman's clock lattice as a graph of perfect matchings: a formula for its height

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Outline

1. Translating a knot into a graph
   - Background from Knot Theory
   - The balanced overlaid Tait graph Γ
   - An example and applications
2. Properties of Γ and the graph $\mathcal{G}$ of perfect matchings
   - The Periphery Proposition and other properties of Γ
   - The graph $\mathcal{G}$ as Kauffman's clock lattice L
   - Main Results
3. Proofs
   - Partition Theorem
   - $\widehat{0}, \widehat{1}$ Theorem
   - Diameter Theorem

Combinatorics and Topology

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Translate a knot into a simple combinatorial object, employ combinatorial techniques, and translate back.

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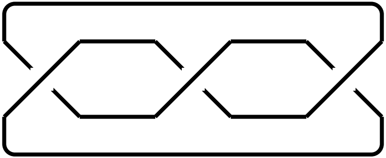

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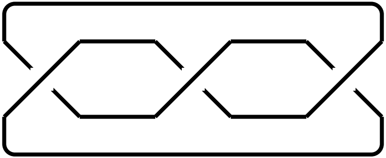

Theorem: (Reidemeister 1926)

Two diagrams represent the same knot ⇔ there is a sequence of Reidemeister moves taking one to the other.

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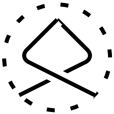
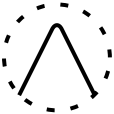
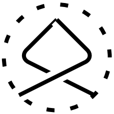
RMI:  $\leftrightarrow$  $\leftrightarrow$ 

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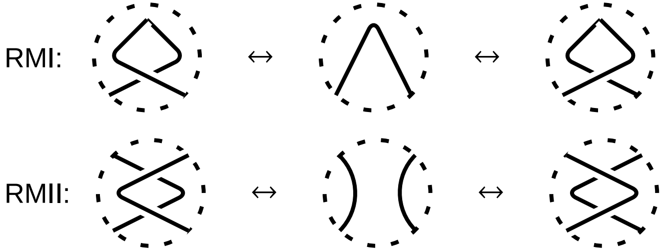

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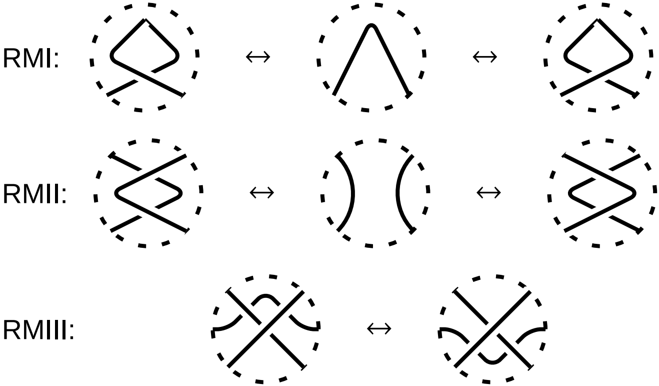

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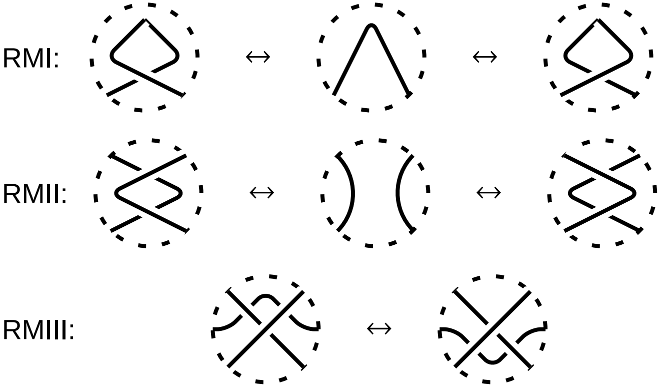

A ***knot invariant*** is an evaluation on a knot diagram that is constant under each of the three *Reidemeister moves*.

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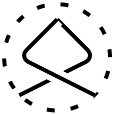
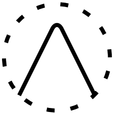
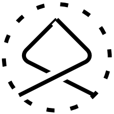
RMI:  ↔  ↔ 

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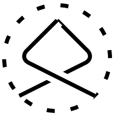
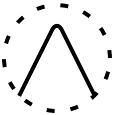
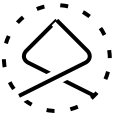
RMI:  ↔  ↔ 

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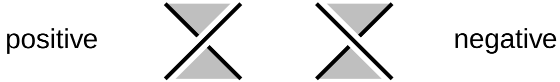

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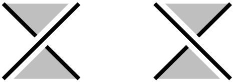
positive  negative

Note that the dual G^* comes from the uncolored regions.

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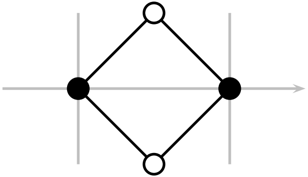

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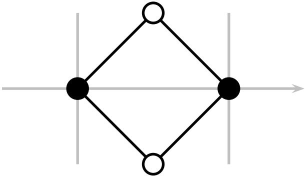

Each face in the overlaid Tait graph $\widehat{\Gamma}$ is a square.

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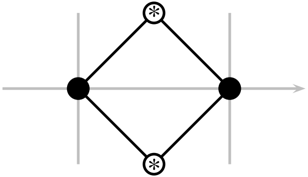

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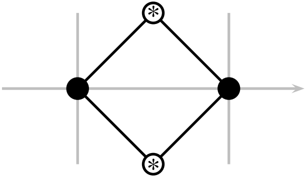

"Balanced" means the two vertex sets are the same size.

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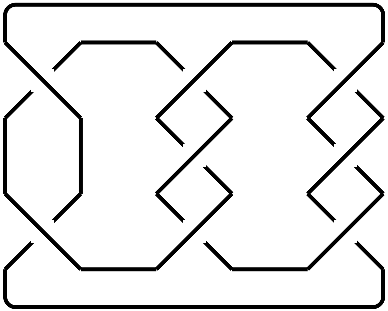

The knot 8_{19} as the $(-2, 3, 3)$-pretzel knot,

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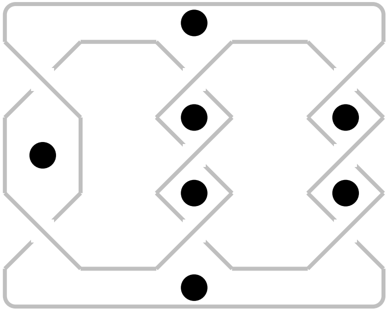

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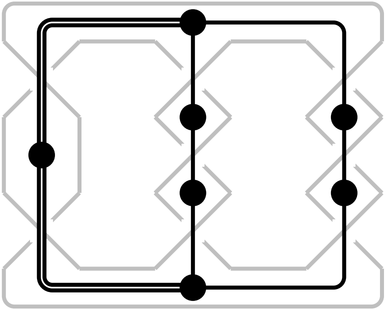

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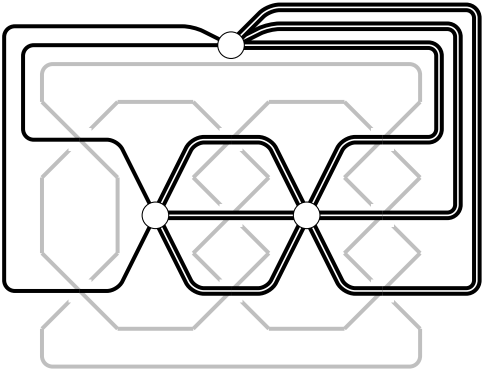

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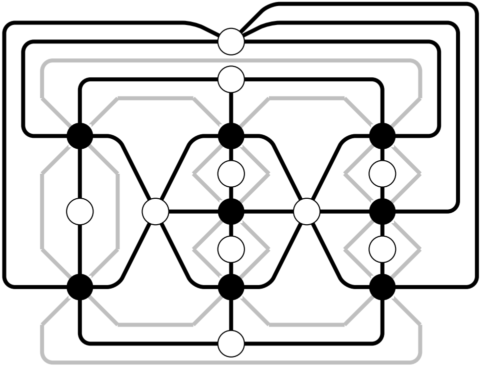

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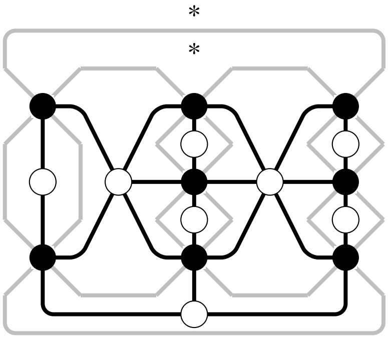

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- (using p-lifts) the twisted Alexander polynomial of a knot together with a representation (C-Dasbach-Russell [CDR12])

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Applications:

- ◊ (Huggett-Mofatt-Virdee) $\widehat{\Gamma}$ to study ribbon graphs from cables
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- (Future work) Perfect matching models for knot homologies.

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$2|E| = 4(|F| - 1) + (1)(2(n_2 + n_3))$ $\Rightarrow n_2 = 2$. □

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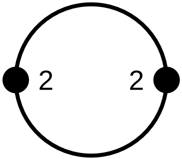

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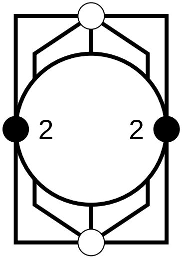

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A knot diagram D is ***prime-like*** if when $D = D_1 \# D_2$, some D_i has no crossings.

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Theorem:

The balanced overlaid Tait graph Γ for a prime-like knot diagram with no nugatory crossings is an elementary graph.

Γ is 2-connected

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A graph Γ is said to be ***n-extendable*** if it is connected, has a set of n independent lines, and every set of n independent lines in Γ extends to (i.e. is a subset of) a perfect matching of Γ.

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Kauffman studied a similar object to obtain $\Delta_K(t)$:

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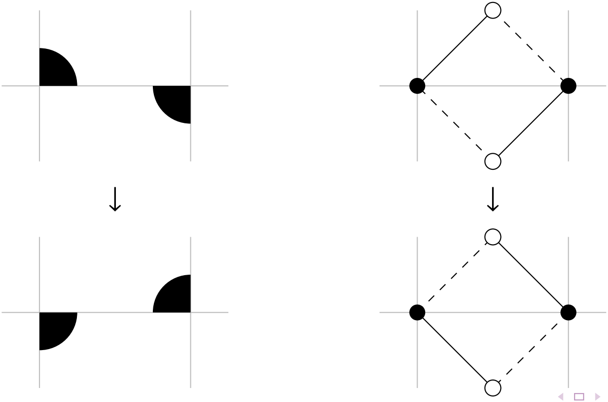
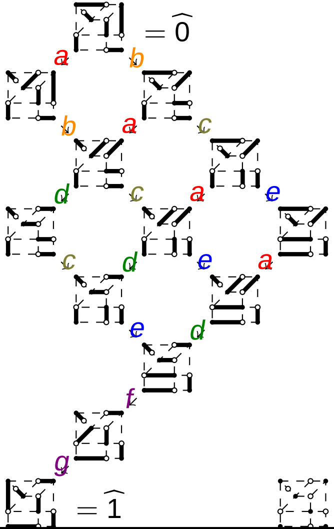
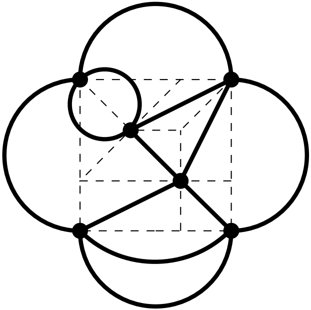
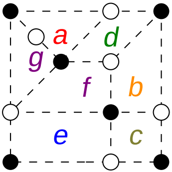


An example of $\mathcal{G}$ from Abe







The graph $\mathcal{G}$ of perfect matchings

Theorem: (Kauffman, Clock Theorem 2.5.) [Kau83]

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Hence any two states in δ are connected by a series of state transpositions.

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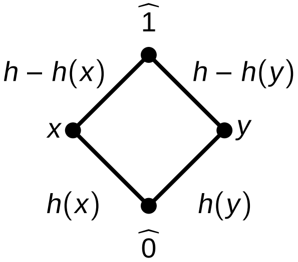

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Theorem: (Abe 2011) [Abe11]

$p(K) \geq c(K)$, the crossing number of K
with equality if and only if K is a 2-bridge knot.

Partitioning the vertices of Γ

Main construction idea:

Partition the vertices of the balanced overlaid Tait graph Γ into leaves $\ell \in \mathcal{L}$ and cycles C_i.

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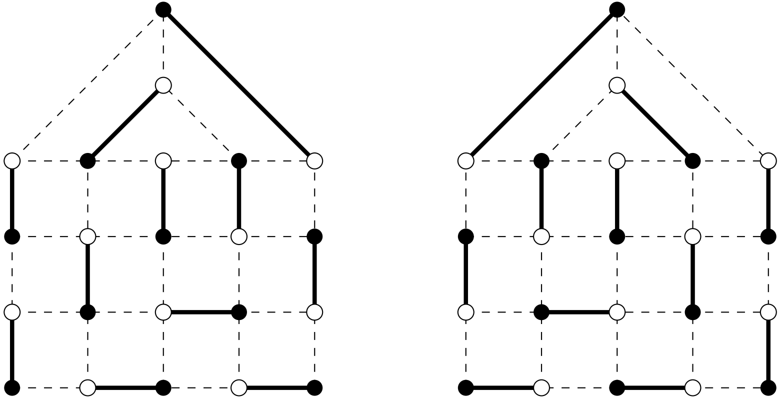
Remark:

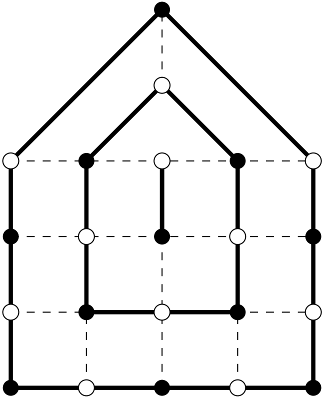
These cycles C_i emerge when the symmetric difference is taken of $\widehat{0}$ and $\widehat{1}$ in Kauffman's clock lattice L!

Partitioning the vertices of Γ



The clocked and counterclocked states of a diagram for $K11n157$.

Partitioning the vertices of Γ



The superposition of these two states of a diagram for $K11n157$.

Partitioning the vertices of Γ

Partition Theorem:

Consider the balanced overlaid Tait graph Γ for a prime-like knot diagram with no nugatory crossings. Then the vertices can be partitioned into leaves $\ell \in \mathcal{L}$ and cycles C_i, where each cycle C_i satisfies the Periphery Proposition and where each interior graph Γ_i is elementary and 2-connected.

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$\widehat{0}, \widehat{1}$ ***Theorem:***

Each C_i is $(\widehat{0}, \widehat{1})$-alternating.

Furthermore, the leaves appear in both of these states.

Properties of the balanced overlaid Tait graph Γ

Diameter Theorem:

Consider the balanced overlaid Tait graph Γ, and

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Then

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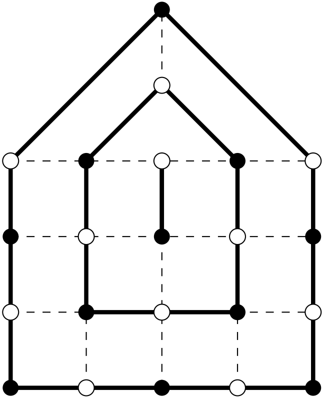
An example using the diameter theorem



The height of the lattice is $15 + 5 = 20$.

Proof of the Partition Theorem

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The periphery $C = C_1$ on the infinite face already satisfies the Periphery Proposition, and the interior graph $\Gamma = \Gamma_1$ is both elementary and 2-connected.

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If C'_i has several components, treat each $C'_i, C'_{i+1}, \ldots$ separately.
If C'_i is a single cycle with no cutvertices, $C'_i = C_i$ and $\Gamma'_i = \Gamma_i$.

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Suppose v is incident with a leaf. Delete all edges incident with v. Collect all of the leaves pruned in the set $\mathcal{L}_{i-1}$.

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"Breaking" cutvertices:

Suppose the deletion of v results in several components, each of which contains a cycle (with v).

Also suppose there is exactly one component C^{odd} that has an odd number of vertices (not including v).

Delete all edges incident with v except for those in C^{odd}.

Need to show there is exactly one odd component

Definition:

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Theorem: (Tutte 1947)

A graph G has a perfect matching $\Leftrightarrow$ the number of odd components of $G - S$ is $|S|$ for all $S \subset V$.

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μ restricts to the remaining graph (G) with $S = \{v\}$,

and so there is exactly one odd component. $\square$

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Turn Γ_i into a knot diagram D_i; it is nugatory and prime-like.

Apply earlier result. □

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Notation:

Decompose C_i into perfect matchings on the cycle subgraph:

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This can only occur when edges e_i and e_j

(from $\circ$ to $\bullet$ on the boundary of the same square face f)

belong to μ_i^0.

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If both e_i and e_j are leaves, f becomes a new cycle C_{ij}. $\square$

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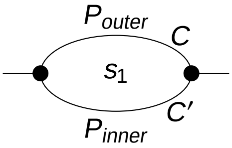

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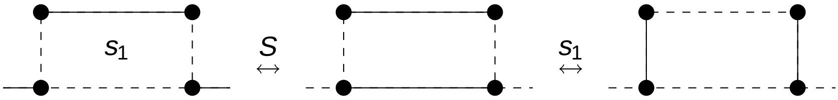

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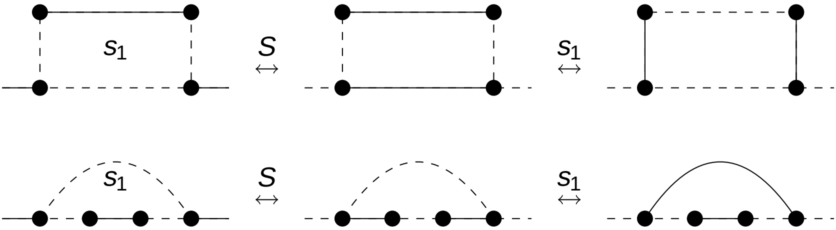

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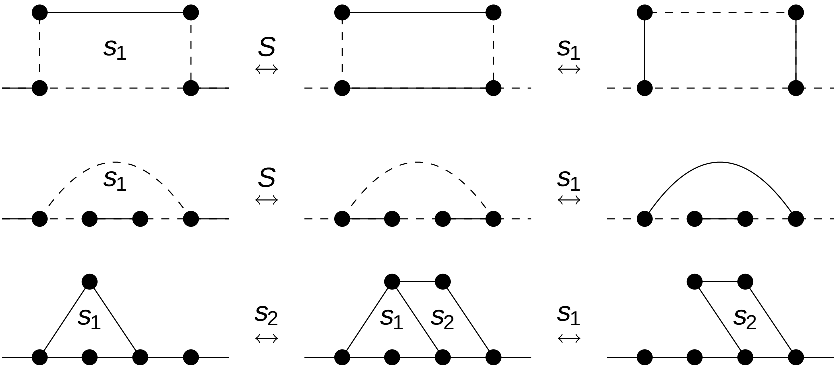

Reduction Moves

Simply Connected Reduction Move:

Removes a simply connected region following the proof above.

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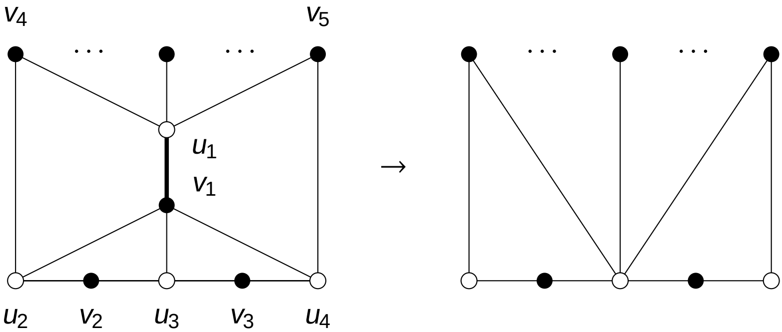

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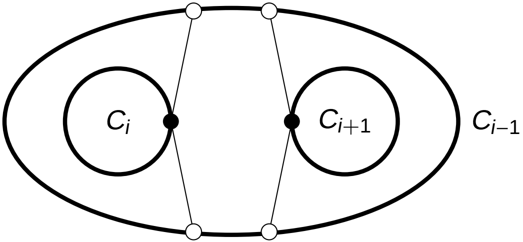
An ***accordion*** joins two disconnected cycles when C_{i-1} is deleted from Γ_{i-1}.



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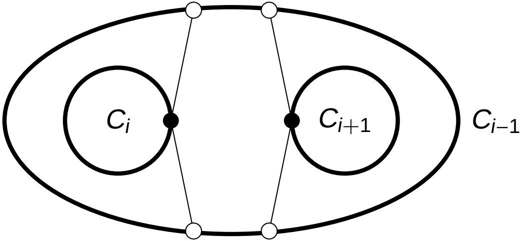
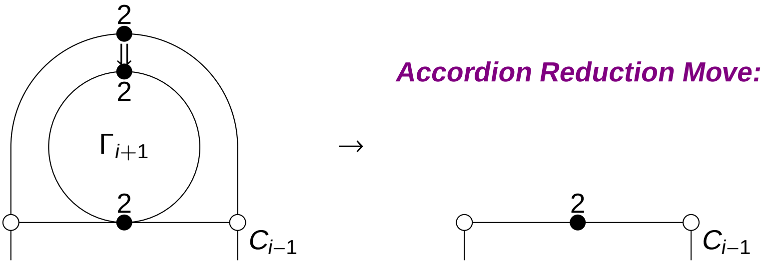


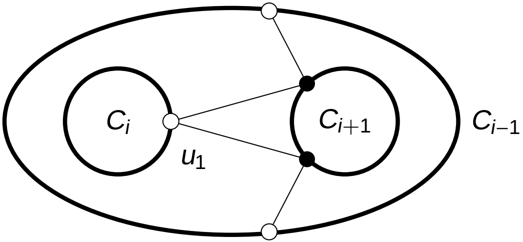


Accordion Reduction Move:

Extra Cycle Reduction Moves

Remove additional cycles (beyond C_i) within a single C_{i-1}.

A ***party hat*** joins cycles separated by cutvertices when C_{i-1} is deleted from Γ_{i-1}.



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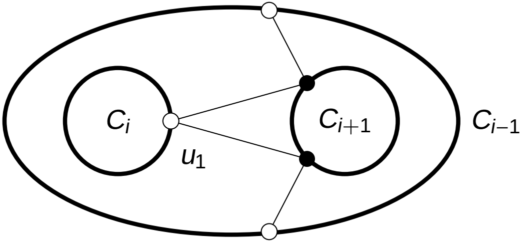

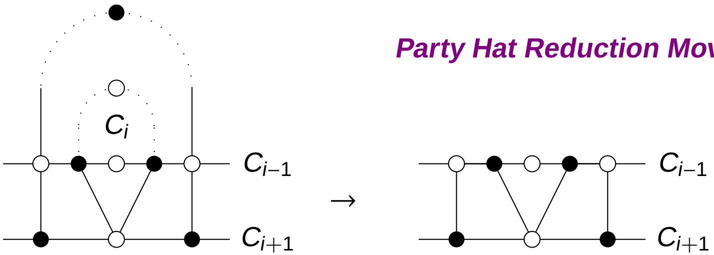

Party Hat Reduction Move:

Induction Step: Flipping a single annulus

Induction Step Lemma:

Flipping all the square faces in $\Gamma_{i-1} \backslash \Gamma_i$ exactly once takes the local perfect matchings of μ^0_{i-1} and μ^1_i to those of μ^1_{i-1} and μ^0_i.

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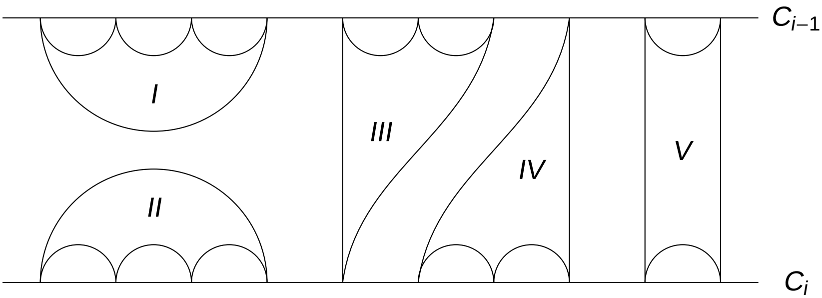

Conclusion

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What else can we learn from the structure of this graph?

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Conjecture:

The number of cycles is related to the ***bridge number*** of the diagram.

This is reinforced by work of Koseleff-Pecker on Chebyshev knots.

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