6 reasons to love Chebyshev knots and billiard table diagrams

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Knot Theory and Its App's to Physics and Quantum Computing, January 6th-9th, 2015

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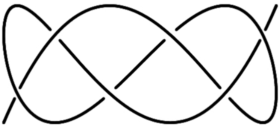

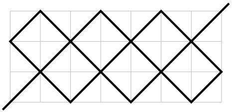

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Knot Theory and Its App's to Physics and Quantum Computing,
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Outline

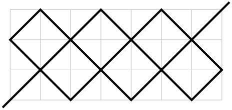
1-3. Chebyshev knots $T(a,b)$ and billiard table diagrams

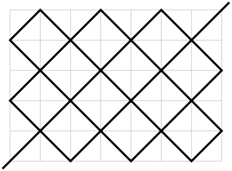
- ***[Koseleff-Pecker '11]***

  Present any k-bridge knot by $T(a,b)$ for $a = 2k - 1$

4-5. Grid diagrams and grid graphs

6. ***[C. '14]*** Jones polynomials of $T(3,b)$ and $T(5,b)$



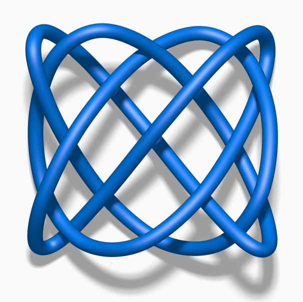


- Intuition (and pictures) behind the heavy notation

A ***Lissajous knot*** is of the form

$$x = \cos(\eta_x t + \phi_x),$$
$$y = \cos(\eta_y t + \phi_y),$$
$$z = \cos(\eta_z t + \phi_z),$$

where $t, \phi_i \in \mathbb{R}$,
$\eta_i \in \mathbb{Z}$ are coprime.



Not all knots are Lissajous, however,
e.g. torus knots and the figure eight.

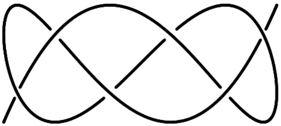
Studied by, e.g., ***[Bogle-Hearst-Jones-Stoilov '94]***, ***[Jones-Przytycki '98]***, ***[Przytycki '98]***.

1. Parametrizations

A ***harmonic knot*** is of the form $x = T_a(t)$, $y = T_b(t)$, $z = T_c(t)$,

where $t \in \mathbb{R}$, $a, b, c \in \mathbb{Z}$ are coprime ***[Comstock 1897]*** , and

$T_n(\cos t') = \cos(nt')$ is the n-th Chebyshev poly (first kind).



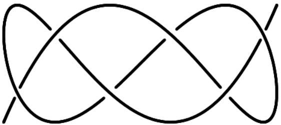
Not all knots are harmonic, however.

Studied by ***[Koseleff-Pecker '12]***

A ***Chebyshev knot*** is $x = T_a(t)$, $y = T_b(t)$, $z = T_c(t+\varphi)$,

where $t \in \mathbb{R}$, $a, b, c \in \mathbb{Z}$ are coprime, $\varphi \in \mathbb{R}$ a constant, and

$T_n(\cos t') = \cos(nt')$ is the n-th Chebyshev poly (first kind).



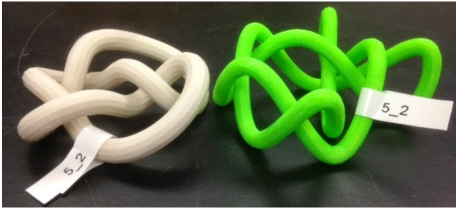
Proposition [Koseleff-Pecker '11]:

All knots are Chebyshev. (More on this later.)

A ***billiard knot*** is the trajectory of a ball traveling in a 3D domain at a straight line, reflecting perfectly off the walls at rational ∠.

Proposition [Jones-Przytycki '98]:

Lissajous knots are precisely the billiard knots in a cube.

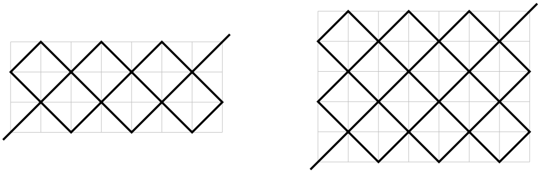


MakerHome: One 3D print every day from home, for a year

Monday, May 26, 2014: Day 273 - Lissajous conformation of 5_2.

Chebyshev knots are examples of generalized billiard knots
in an $a \times b \times c$ rectangular prism.

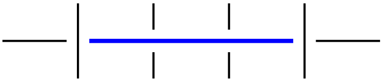
These can be projected onto ***billiard table diagrams T(a,b)***.



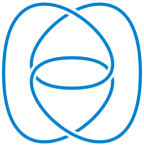
To simplify, we replace c, ϕ
with a string of $+, -$ corresponding to the crossings.

3. Natural indexing

A ***bridge*** is one of the arcs in a knot diagram.



The ***bridge index*** $br(K)$ of a knot K is the minimum number of disjoint bridges which together include all over-crossings



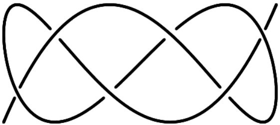
OR equivalently the minimum number over all diagrams of local maxima of the knot diagram taken with a Morse function.

3. Natural indexing

***Proposition* *[Koseleff-Pecker '11]*:**

For knot K and $br(K) \leq m \in \mathbb{N}$, K is some $T(a, b)$

where $a = 2m - 1$ and $b \equiv 2 \pmod{2a}$.

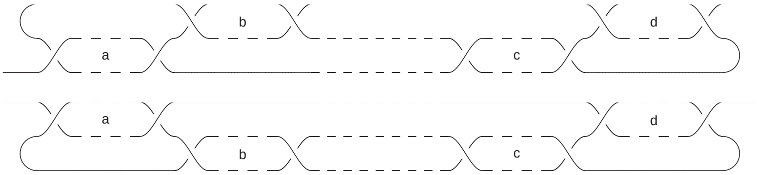


***Theorem* *[Koseleff-Pecker '11]*:**

Every knot has a projection that is a Chebyshev plane curve.

3. Natural indexing

The class of ***2-bridge knots*** can be completely described using Conway notation [a,b,...,c,d].

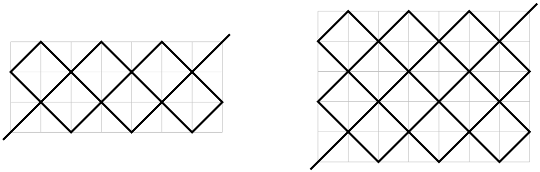


Because one can associate to this sequence a continued fraction, they are also called ***rational knots***.

$$\frac{p}{q} = \frac{1}{a + \frac{1}{b+\frac{1}{\dots+\frac{1}{c+\frac{1}{d}}}}}$$

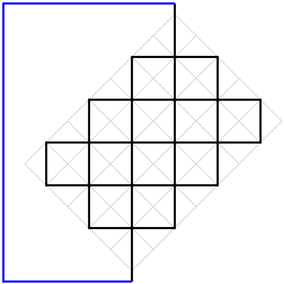
3. Natural indexing

We can restrict our attention to $T(3, b)$ to consider all ***2-bridge knots*** together with the unknot



and $T(5, b)$ to consider all ***3-bridge knots*** together with all 2-bridge knots and the unknot.

Rotate 45° and close the ends to obtain a ***grid diagram***, where the crossings are not necessarily right:

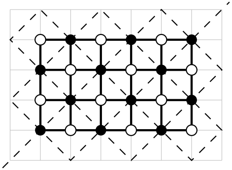


Fix the crossings and rotate back to obtain a ***Legendrian knot***.

5. Alexander polynomial via grid graphs

The ***"dimer" graph*** ***[C.-Dasbach-Russell '14]*** for a billiard table diagram is the popular ***grid graph*** from graph theory.

Dimer or perfect matching models on these grid graphs also appear in statistical mechanics.



The ***Alexander polynomial*** can be swiftly computed from this ***[C.-Dasbach-Russell '14]***.

Consider $T(a, b)$ with $a = 3$ or $a = 5$ and with b coprime.

Order the N crossings lexicographically.

Obtain a knot from a string of $\{+, -\}$ of length N.

Goal [C. ’14]:

To compute the Jones polynomials directly from the string.

Want a notation that is sensitive to whether a crossing is $\pm$.

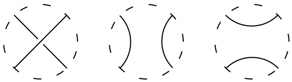
Let f_b be the ***Kauffman bracket polynomial*** $\langle T(3, b)\rangle$

and h_b $\langle T(5, b)\rangle$.

We compute writhe at the end.

Apply the ***unoriented Skein relation***

$$\langle L \rangle = A\langle L_0 \rangle + A^{-1}\langle L_\infty \rangle$$



Notation:

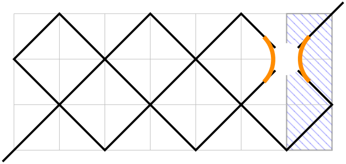
To each crossing assign some monomial:

If the $\pm$ crossing is smoothed vertically, use $A^{\pm}$.

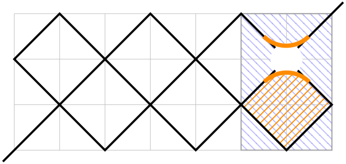
If the $\pm$ crossing is smoothed horizontally, use $A^{\mp}$.

If the $\pm$ crossing is resolved by Reidemeister move I, use $f_2^{\pm}$

6. Jones polynomials of 2-bridge knots / Proof Idea

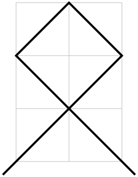


$(\ldots, A^{\pm})$

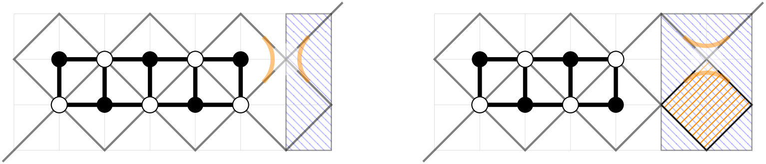


$(\ldots, f_2^{\mp}, A^{\mp})$

where $f_2^{\pm} = -A^{\mp 3}$ is the Kauffman bracket polynomial of $T(3,2)$



Look at the “dimer graph” again to see the $2 \times b$ grid graph.



The smoothings relate to the $2 \times (b-1)$ and $2 \times (b-2)$ grid graphs.

Let $C = [A^{\pm}, A^{\pm}] + [f_2^{\mp}, A^{\mp}]$ and substitute C from the left.

Theorem [C.]:

The Kauffman bracket polynomials f_b of $T(3, b) \supseteq \{\text{2-bridge knots}\}$ obey the following recursion rules:

If a summand in f_{b-1} ends in ____ then it is a summand in f_b:

$(..., A^{\pm})$	with a C replacing the $A^{\pm}$
$(..., [f_2^{\mp}, A^{\mp}])$	ending with $A^{\pm}$
$(..., C)$	with $[C, A^{\pm}] + [A^{\pm}, f_2^{\mp}, A^{\mp}]$ replacing the C.

Proposition [C.]:

The # of terms in the expansion of f_b is
(a sequence that is an offset by four of)

$$a(0) = 1,$$

$$a(1) = a(2) = 0,$$

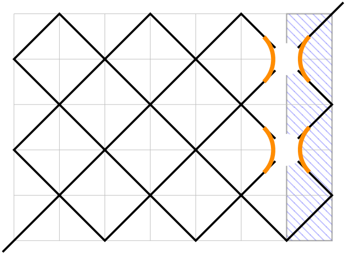
$$a(n) = a(n-2) + a(n-3),$$

the ***Padovan sequence***,

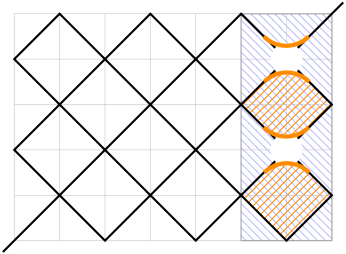
[A000931] in The On-Line Encyclopedia of Integer Sequences.

6. Jones polynomials of 3-bridge knots / Proof Idea

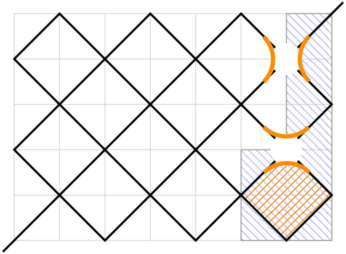
Two of the four terms are just as before:



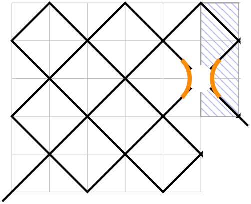
$$(\ldots, A^{\pm}, A^{\pm})$$

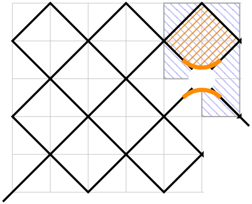


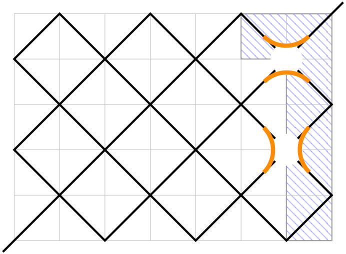
$$(\ldots, f_2^{\mp}, f_2^{\mp}, A^{\mp}, A^{\mp})$$



The third term can itself be reduced:

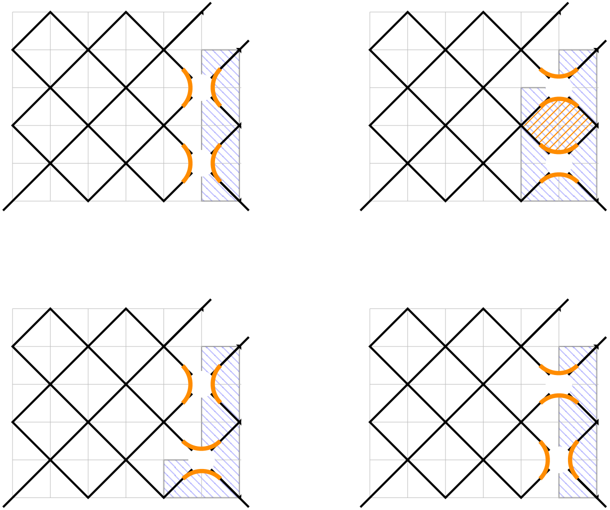






The fourth term gives a 2-tangle, but it can also itself be reduced.

Keep track of the two pairs of ends.



Trick: these last two are really rectangles!

Let $\mathcal{P}_n$ be the set of all partitions of the integer n into blocks P_i.

Recall $\delta = (-A^2 - A^{-2})$ and $f_2^{\pm} = -A^{\mp 3}$ as above.

Set $X = \delta[A^{\pm}, A^{\pm}] + [A^{\pm}, A^{\mp}] + [A^{\mp}, A^{\pm}]$.

Let $P_1 = [A^{\pm}, A^{\pm}]$,

$$P_2 = [f_2^{\mp}, f_2^{\mp}, A^{\mp}, A^{\mp}] + [A^{\pm}, f_2^{\mp}, A^{\pm}, A^{\mp}] + \delta[A^{\pm}, A^{\pm}, A^{\mp}, A^{\pm}] + [A^{\pm}, A^{\mp}, A^{\mp}, A^{\pm}] + [A^{\mp}, A^{\pm}, A^{\mp}, A^{\pm}],$$

$$P_i = \begin{cases} [f_2^{\mp}, A^{\pm}, A^{\mp}, A^{\mp}, [f_2^{\mp}, A^{\mp}, A^{\mp}, A^{\mp}]^j, A^{\pm}, A^{\mp}] \\ \quad + [A^{\pm}, [f_2^{\mp}, f_2^{\mp}, A^{\mp}, A^{\mp}]^{j+1}, A^{\pm}] & \text{for } i = 2j+3 \text{ and} \\ [A^{\pm}, [f_2^{\mp}, f_2^{\mp}, A^{\mp}, A^{\mp}]^j, f_2^{\mp}, A^{\pm}, A^{\mp}] \\ \quad + [X, [f_2^{\mp}, A^{\mp}, A^{\mp}, A^{\mp}]^j, A^{\mp}, A^{\pm}] & \text{for } i = 2j+2. \end{cases}$$

* White lie: See P_i' vs. $\widetilde{P}_i'$ defined in my paper. Q_i is similar.

Main Theorem [C.]:

The Kauffman bracket polynomials h_b of $T(5,b) \supseteq \{$3-bridge knots$\}$:

$$\begin{aligned} h_1 &= 1 \\ h_2 &= (A^{\pm}, A^{\pm}) + \delta(A^{\pm}, A^{\mp}) + \delta(A^{\mp}, A^{\pm}) + \delta^2(A^{\mp}, A^{\mp}) \\ h_3 &= (h_2, A^{\pm}, A^{\pm}) + (f_2^{\mp}, f_2^{\mp}, A^{\mp}, A^{\mp}) + (g_2, A^{\pm}, A^{\mp}) + (f_2^{\mp}, f_2^{\pm}, A^{\mp}, A^{\pm}) \end{aligned}$$

and for $b \geq 4$,

$$h_b = \sum_{i=3}^{b-1} ([h_3, P_{i-2}] + [h_2, P_{i-1}] + [Q_i], \mathcal{P}_{b-1-i}).$$

Proposition [C.]:

The # of terms in the expansion of h_b is 2^{b-4}.

Remark:

Note that this is far fewer than the usual $2^{2(b-1)}$ terms using the Skein relation for the $2(b-1)$ crossings of the diagram.

Property [C.]:

The writhe $w(T(3, b+3)) =$

$$\begin{cases} w(T(3,b)) + (\pm 1) + (\mp 1) + (\pm 1) & \text{when } b \equiv 1 \mod 3 \text{ and} \\ w(T(3,b)) + (\pm 1) + (\pm 1) + (\mp 1) & \text{when } b \equiv 2 \mod 3, \end{cases}$$

corresponding to the signs of the three last crossings.

Property [C.]:

The writhe $w(T(5, b+5)) =$

$$\begin{cases} w(T(5,b)) + (\pm 1) + (\pm 1) + (\mp 1) + (\pm 1) & \\ \quad +(\mp 1) + (\mp 1) + (\mp 1) + (\pm 1) + (\pm 1) + (\pm 1) & b \equiv 1 \bmod 5 \text{ and } 2|b, \\ w(T(5,b)) + (\pm 1) + (\pm 1) + (\pm 1) + (\mp 1) & \\ \quad +(\mp 1) + (\mp 1) + (\pm 1) + (\mp 1) + (\pm 1) + (\pm 1) & b \equiv 1 \bmod 5 \text{ and } 2 \nmid b, \\ \quad \vdots & \vdots \\ w(T(5,b)) + (\pm 1) + (\pm 1) + (\pm 1) + (\pm 1) & \\ \quad +(\pm 1) + (\mp 1) + (\mp 1) + (\mp 1) + (\pm 1) + (\mp 1) & b \equiv 4 \bmod 5 \text{ and } 2|b, \\ w(T(5,b)) + (\pm 1) + (\pm 1) + (\pm 1) + (\pm 1) & \\ \quad +(\mp 1) + (\pm 1) + (\mp 1) + (\mp 1) + (\mp 1) + (\pm 1) & b \equiv 4 \bmod 5 \text{ and } 2 \nmid b, \end{cases}$$

corresponding to the signs of the ten last crossings.

Conclusion

Reason 7:

Useful for random knots – see my talk ***[JMM, Tuesday 2pm 006D]***

Thanks:

to Pierre-Vincent Koseleff as well as to

TOPOSYS

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