Dimer Models for the Alexander and Twisted Alexander Polynomials of Knots

Moshe Cohen
(Joint with Oliver Dasbach, Heather M. Russell)

Department of Mathematics and Computer Science
Bar-Ilan University
www.math.biu.ac.il/∼cohenm10/

11th Haifa Workshop on Interdisciplinary Applications of Graph Theory, Combinatorics and Algorithms,
May 18th, 2011

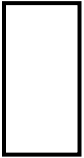
Outline

Warmup

Question: How many different ways can we tile a $2 \times n$ gameboard with 2×1 dominos?

Warmup

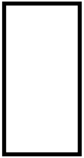
Question: How many different ways can we tile a $2 \times n$ gameboard with 2×1 dominos?



: $n = 1$

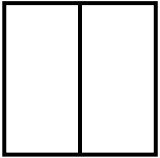
Warmup

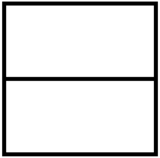
Question: How many different ways can we tile a $2 \times n$ gameboard with 2×1 dominos?



: $n = 1$

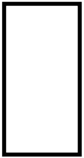
$n = 2$:





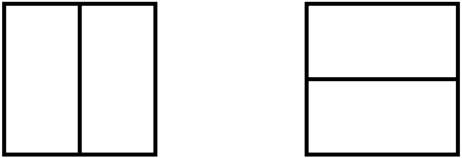
Warmup

Question: How many different ways can we tile a $2 \times n$ gameboard with 2×1 dominos?

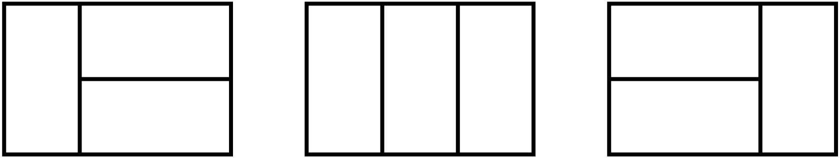


$: n = 1$

$n = 2 :$



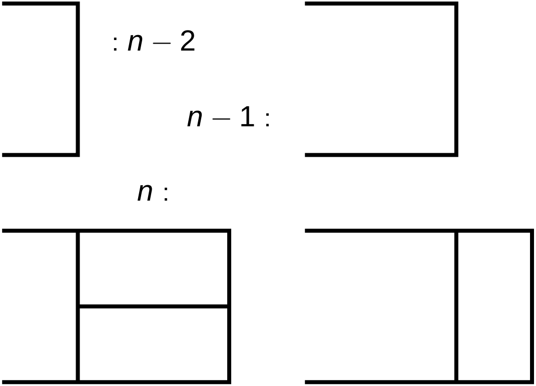
$n = 3 :$



Warmup

Answer: The number of ways $f(n)$ satisfies

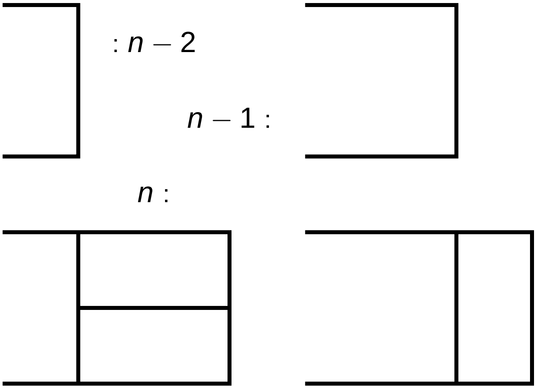
$$f(n-2) + f(n-1) = f(n).$$



Warmup

Answer: The number of ways $f(n)$ satisfies

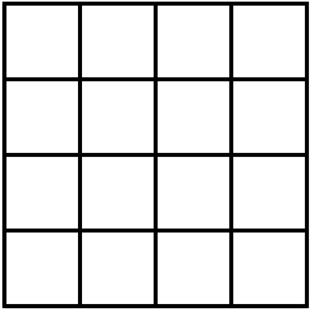
$$f(n) = f(n-2) + f(n-1).$$



Thus we get $f(n)$ is the n-th Fibonacci number!

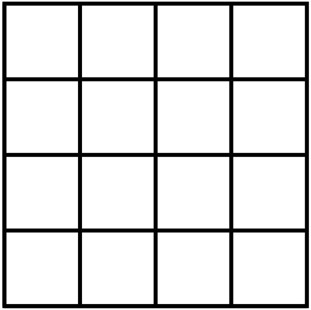
From combinatorics to statistics

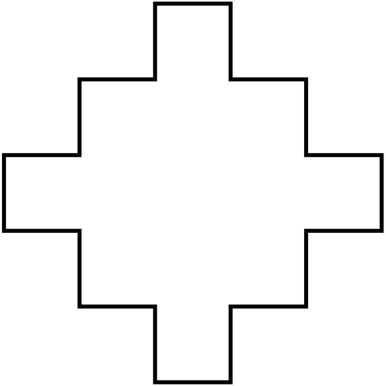
Combinatorialists ask how many ways to domino tile general gameboards



From combinatorics to statistics

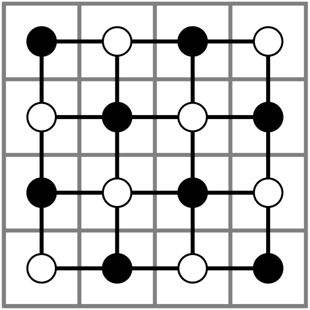
Combinatorialists ask how many ways to domino tile general gameboards (and aztec diamonds).





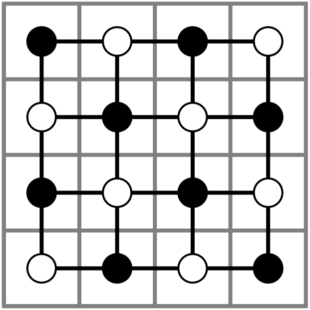
From combinatorics to statistics

Graph theorists look at the perfect matchings of associated bipartite graphs.



From combinatorics to statistics

Graph theorists look at the perfect matchings of associated bipartite graphs.



A ***perfect matching*** μ

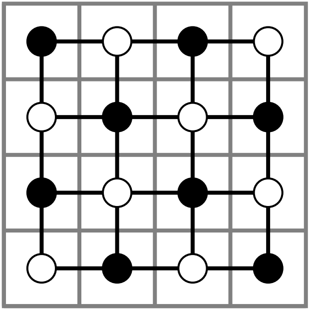
is a collection

of non-incident edges

that covers the graph.

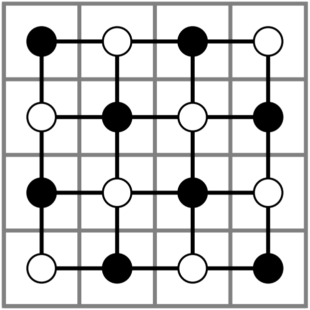
From combinatorics to statistics

Physicists in statistical mechanics ask about the probability of achieving certain states.



From combinatorics to statistics

Physicists in statistical mechanics ask about the probability of achieving certain states.



A ***dimer*** is just an edge.

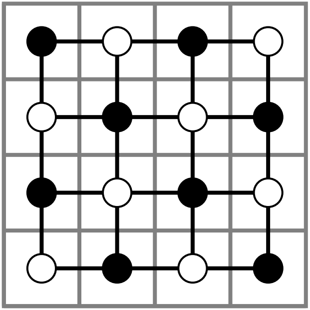
A ***dimer covering***

is a collection

of non-incident dimers

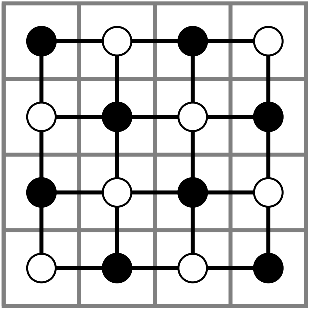
that covers the graph.

From combinatorics to statistics



Now let's ask for the complete list of tilings and not just the number of them.

From combinatorics to statistics



We use an idea from

the generalization of

the Matrix Tree Theorem.

So let's look at

matrices from the graphs.

Now let's ask for the complete list of tilings
and not just the number of them.

Matrices from graphs: the Matrix Tree Theorem

Consider the diagonal matrix D where $d_{ii} = \deg v_i$,

and let A be the adjacency matrix of the graph.

Matrix Tree Theorem (Kirchhoff):

The determinant of any cofactor of the matrix $D - A$ gives the number of spanning trees of the graph.

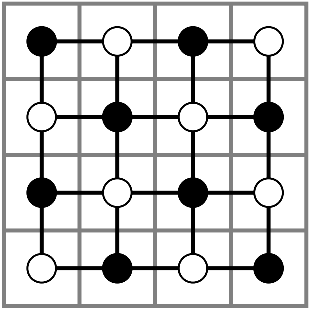
The spanning trees themselves can be recovered by using variables to represent the edges in this matrix.

Matrices from graphs: bipartite adjacency matrix

Consider the adjacency matrix of the given bipartite graph Γ.

$$\left(\begin{array}{c|c} 0 & M \\ \hline M^T & 0 \end{array}\right)$$

Call M the ***bipartite adjacency submatrix***.



$$\begin{pmatrix}
e_{11} & & e_{13} & & & & & \\
e_{21} & e_{22} & & e_{24} & & & & \\
e_{31} & & e_{33} & e_{34} & e_{35} & & & \\
 & e_{42} & & e_{44} & & e_{46} & & \\
 & & e_{53} & & e_{55} & & e_{57} & \\
 & & & e_{64} & e_{65} & e_{66} & & e_{68} \\
 & & & & e_{75} & & e_{77} & e_{78} \\
 & & & & & e_{86} & & e_{88}
\end{pmatrix}$$

Matrices from graphs: determinant and permanent

Recall the determinant of a matrix $M = (m_{ij})$

$$det(M) = \sum_{\sigma \in \mathfrak{S}} \prod_i (-1)^{sign(\sigma)} m_{i\sigma(i)}$$

The ***permanent*** or ***unsigned determinant*** is

$$perm(M) = \sum_{\sigma \in \mathfrak{S}} \prod_i m_{i\sigma(i)}$$

Matrices from graphs: determinant and permanent

Proposition:

The terms in the permanent expansion of a bipartite adjacency submatrix associated with a bipartite graph give the complete list of perfect matchings of the graph.

Proof:

Each term in the permanent expansion is a permutation σ matching each vertex i in the first vertex set to a vertex $\sigma(i)$ in the second vertex set. $\square$

Kasteleyn weighting: signing the edges

A ***Kasteleyn weighting*** of a plane bipartite graph is a signing of the edges such that $\#$ negatives around a particular face is

- odd if the face has length 0 mod 4 or
- even if the face has length 2 mod 4.

Lemma:

Suppose G has a Kasteleyn weighting. Then so does $G \setminus e$.

Kasteleyn weighting: signing the edges

Proposition:

The determinant expansion of a bipartite adjacency submatrix associated with a Kasteleyn-weighted bipartite graph gives the complete list of perfect matchings up to sign.

Proof:

Two permutations differ by a transposition $\longleftrightarrow$
$\exists$ four non-zero terms in a rectangle in the matrix $\longleftrightarrow$
$\exists$ a square face in the graph.
$\exists!$ negative sign in each square, so these have opposite signs in both the matrix and the perfect matching. $\square$

Main Results

Theorem 1:

Let Γ be a balanced overlaid Tait graph associated with K. Then

$$\sum_{m} \prod_{e \in m} \mu(e) \doteq \Delta_K(t),$$

the Alexander polynomial of K up to sign and a power of t.

(over all perfect matchings m and all edges in each).

Main Results

Theorem 2:

Let Γ_ρ be the p-lift obtained from Γ for K and ρ. Then

$$\sum_m \prod_{e \in m} \mu(e) \doteq W_K(t),$$

the twisted Alexander polynomial of K with representation ρ

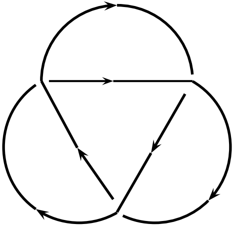
up to sign and a power of t

(over all perfect matchings m and all edges in each).

Background in knot theory

A ***knot*** K is S^1 embedded in $\mathbb{R}^3 \cup \infty$. We ***orient*** the knot.

A ***knot diagram*** D is the projection of the knot onto $\mathbb{R}^2$ with under- and over-crossing information.

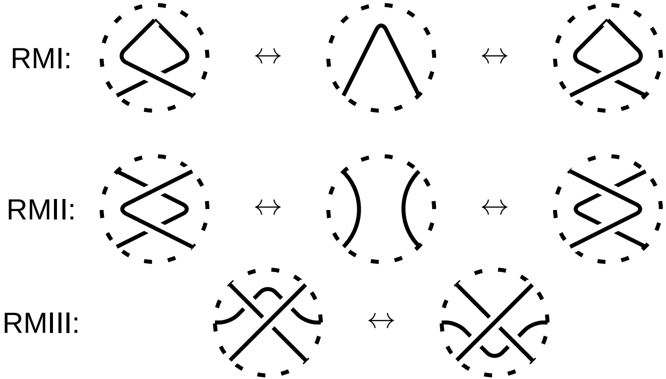


Theorem (Reidemeister 1926):

Two diagrams represent the same knot $\Leftrightarrow$

$\exists$ a sequence of Reidemeister moves taking one to the other.

Background in knot theory



A ***knot invariant*** is an evaluation on a knot diagram that is constant under each of the three *Reidemeister moves*.

Why study knots?

Understanding low-dimensional topology:

- Topological quantum field theory gives 2-manifolds.
- Surgery on knots gives 3-manifolds.
- TQFT again gives 4-manifolds.
- Hyperbolic geometery triangulates knot complements.
- Number theory recognizes these hyperbolic volumes.

By applying techniques from graph theory:

- Need only study plane graphs!
- Nice properties emerge from the four-valent crossings.
- Tables of (small) knot diagrams can be found online.

Graphs from knots: the signed Tait graph G

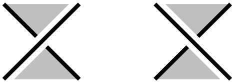
A ***signed graph*** has edges weighted $+1$ or -1.

Checkerboard color the regions of a knot diagram D.

Definition:

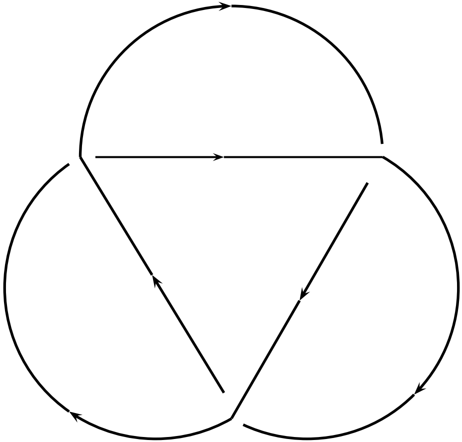
The ***signed Tait graph*** G associated with D has

$V(G) = \{\text{colored regions}\}$ and $E(G) = \{\text{crossings of } D\}$.

positive  negative

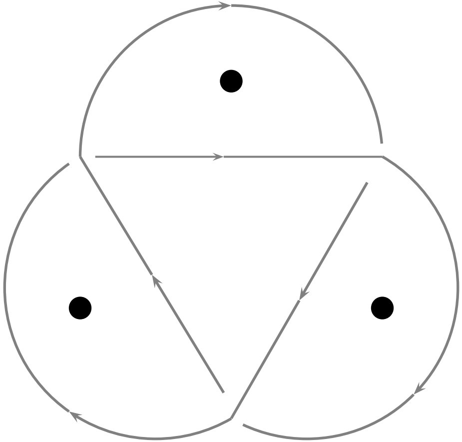
Note that the dual G^* comes from the uncolored regions.

Graphs from knots: the signed Tait graph G



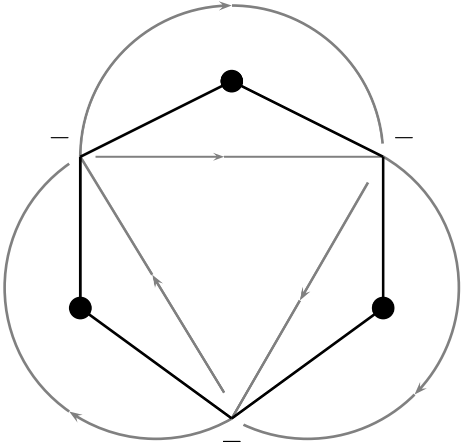
Given a knot diagram,

Graphs from knots: the signed Tait graph G



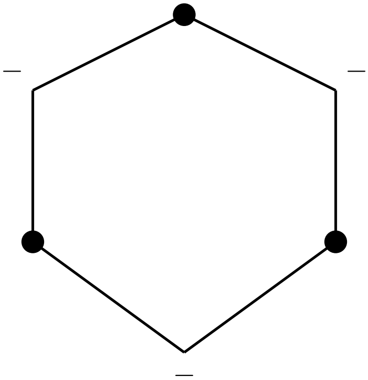
checkerboard color the regions to get vertices

Graphs from knots: the signed Tait graph G



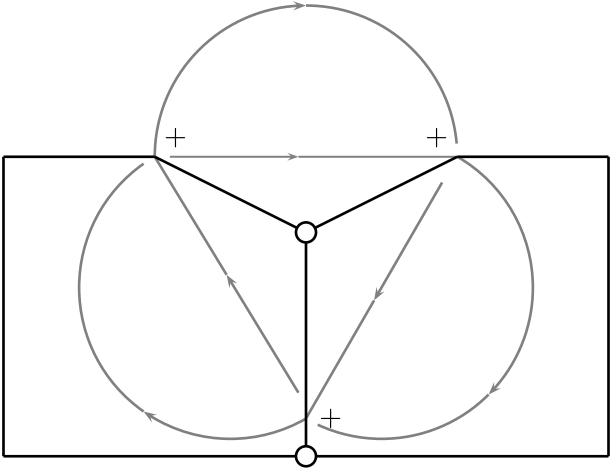
and sign the edges through the crossings

Graphs from knots: the signed Tait graph G



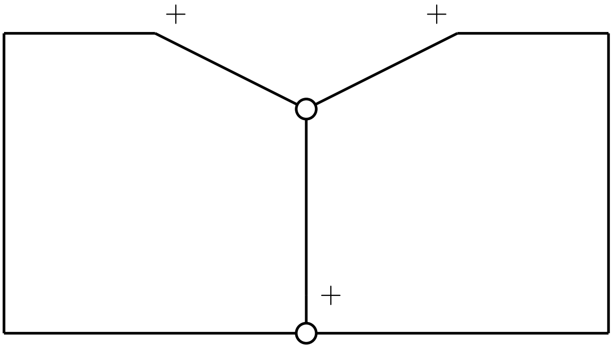
to obtain the signed Tait graph G.

Graphs from knots: the signed Tait graph G



Also use the white regions

Graphs from knots: the signed Tait graph G

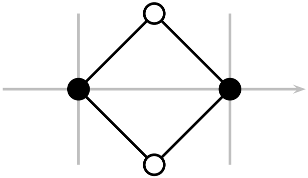


to obtain the signed dual Tait graph G^*.

Graphs from knots: the overlaid Tait graph $\widehat{\Gamma}$

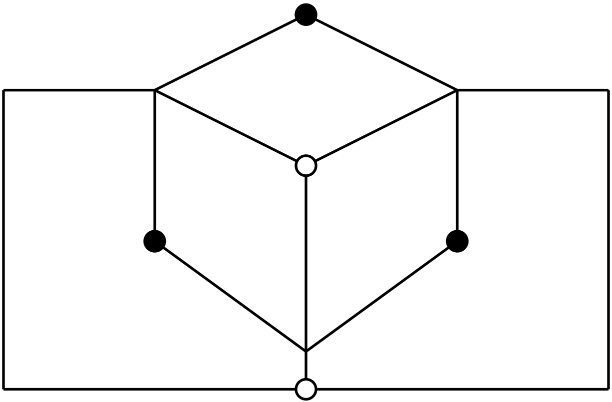
Definition:

The ***overlaid Tait graph*** $\widehat{\Gamma}$ associated with D is bipartite with $V(\widehat{\Gamma}) = [E(G) \cap E(G^*)] \sqcup [V(G) \sqcup V(G^*)]$ and $E(\widehat{\Gamma})$ the half-edges of G and G^*.



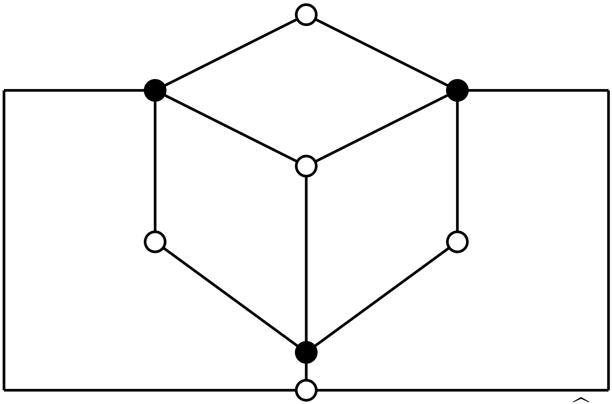
Each face in the overlaid Tait graph $\widehat{\Gamma}$ is a square.

Graphs from knots: the overlaid Tait graph $\widehat{\Gamma}$



Overlay the two Tait graphs

Graphs from knots: the overlaid Tait graph $\widehat{\Gamma}$

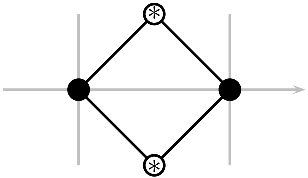


to obtain the (unsigned) overlaid Tait graph $\widehat{\Gamma}$.

Graphs from knots: the balanced overlaid Tait graph Γ

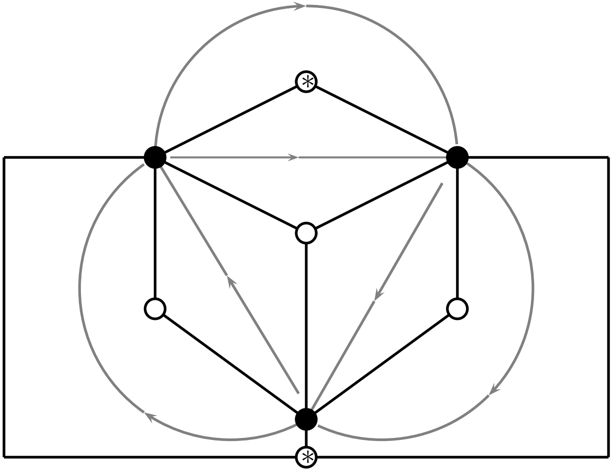
Definition:

The ***balanced overlaid Tait graph*** Γ associated with D is obtained from $\widehat{\Gamma}$ by removing two vertices from the larger set that lie on the same face:



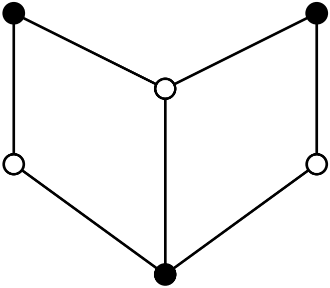
"Balanced" means the two vertex sets are the same size.

Graphs from knots: the balanced overlaid Tait graph Γ



Delete the starred vertices

Graphs from knots: the balanced overlaid Tait graph Γ



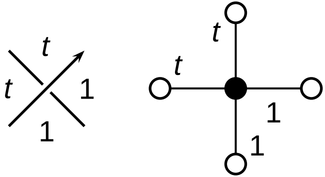
to obtain the balanced overlaid Tait graph Γ.

The Alexander polynomial $\Delta_K(t)$

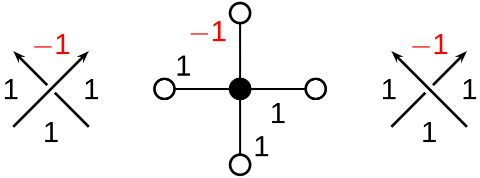
The ***Alexander polynomial*** $\Delta_K(t)$, a Laurent poly in $\mathbb{Z}[t^{\pm\frac{1}{2}}]$, can be defined as the determinant of the *Alexander matrix*.

Compute the *Fox* partial derivatives of the relations with respect to the generators of the *Dehn presentation* for the fundamental group $\pi_1(S^3 - K)$ of the knot complement.

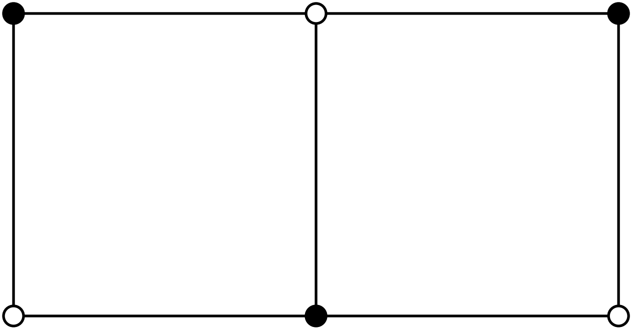
After abelianization, this amounts to the following local information about each crossing:



Furthermore, *Kauffman's trick* of assigning one minus sign at each crossing satisfies *Kasteleyn's weighting*:

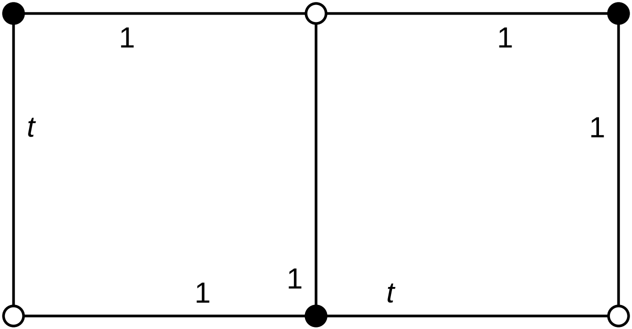


Example: the Alexander Dimer for the Trefoil



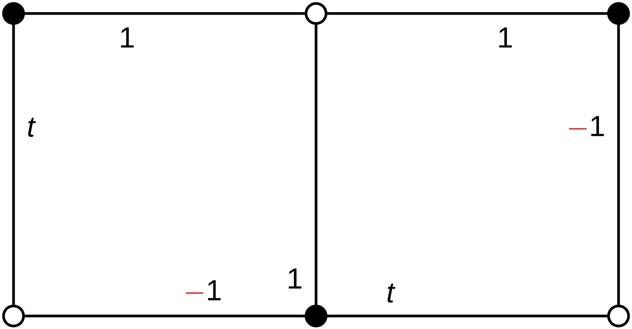
The bipartite graph for the Trefoil.

Example: the Alexander Dimer for the Trefoil



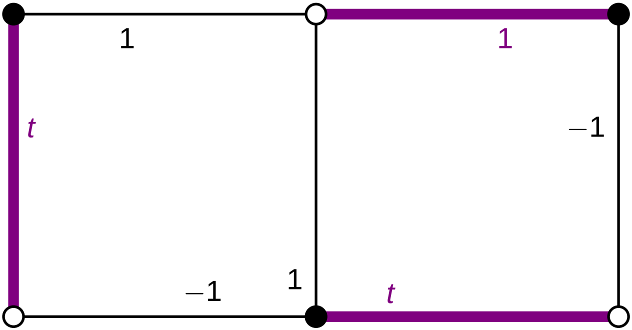
Labeling at each of the crossings.

Example: the Alexander Dimer for the Trefoil



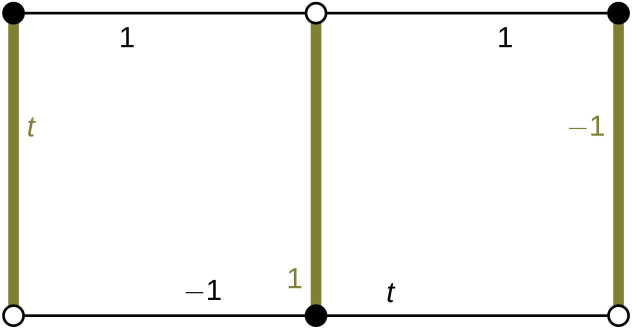
Kauffman's trick to yield a Kasteleyn weighting.

Example: the Alexander Dimer for the Trefoil



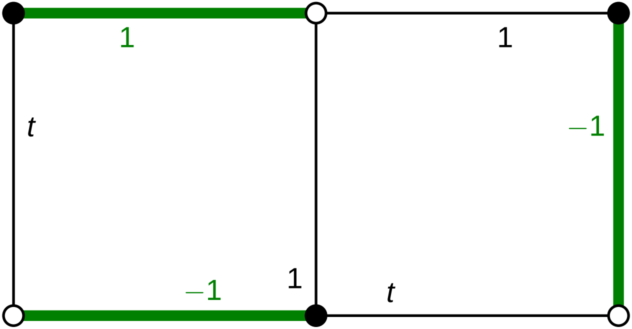
One perfect matching, yielding $(t)(1)(t) = t^2$.

Example: the Alexander Dimer for the Trefoil



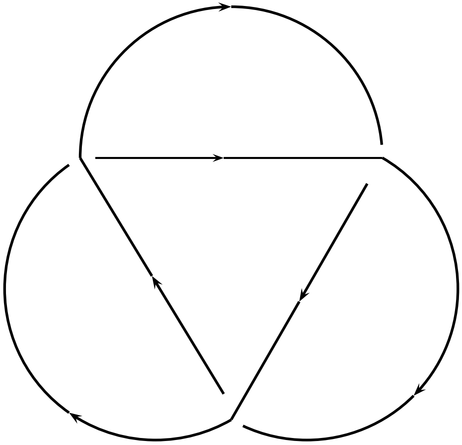
A second perfect matching, yielding $(t)(1)(-1) = -t$.

Example: the Alexander Dimer for the Trefoil



The third and last perfect matching, yielding $(1)(-1)(-1) = 1$.

Example: the Alexander Dimer for the Trefoil



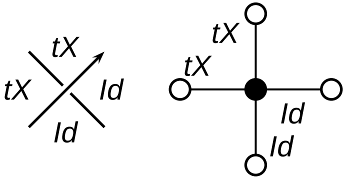
The Alexander polynomial is $\Delta_L(t) = t^2 - t + 1$.

The twisted Alexander polynomial $W_K(t)$

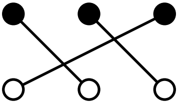
The ***twisted Alexander polynomial*** $W_K(t)$, a poly in $R[t^{\pm 1}]$, can be defined as the determinant of the *Alexander matrix* – after twisting by representation $\gamma : \pi_1(S^3 - K) \to GL_p(\mathbb{Z})$.

Replace each entry of the original *Alexander matrix* with a $p \times p$ block carrying the representation above.

We can use the following local information about each crossing:



Replace each original vertex with *p* copies, and replace each edge with *p* edges, twisting as necessary.



Relating the two dimer models

Suppose the graph Γ is obtained from a knot diagram,
and let Γ_ρ be obtained from Γ together with the representation ρ.

Definition:

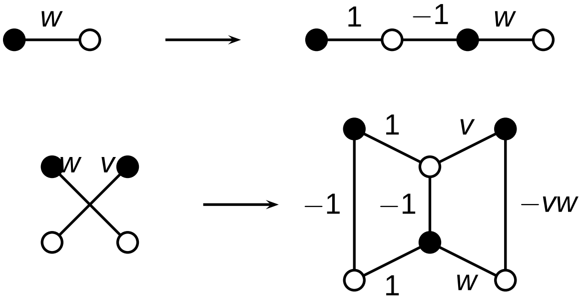
A ***p*-lift** of a graph on n vertices
is a graph on pn vertices with a covering map.

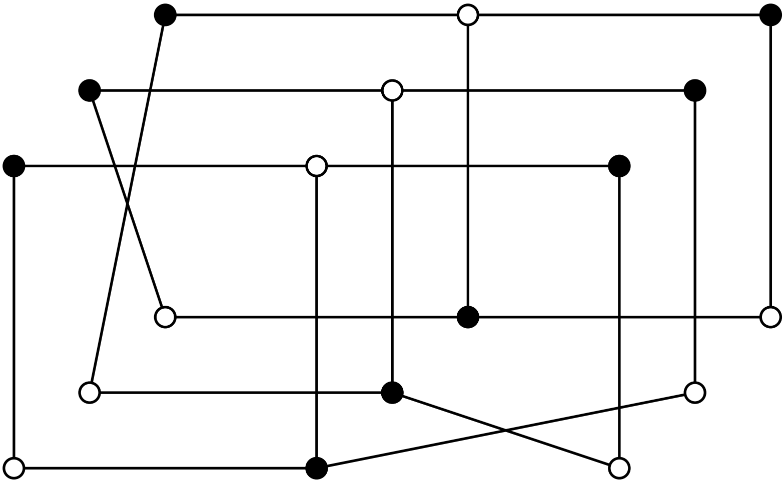
Proposition:

By construction, Γ_ρ is a p-lift of Γ.

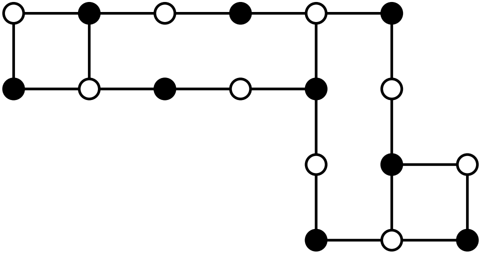
In order to use *Kauffman's trick* as before, we need to take care of crossings coming from the "twists".

We use Kuperberg's *triple splitting* and *butterfly* tricks:

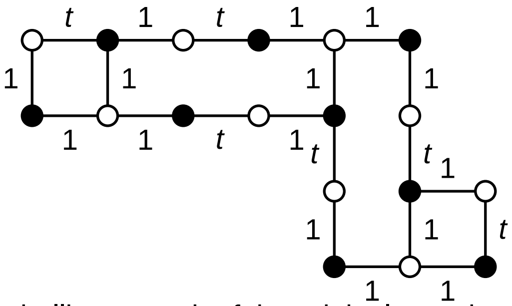




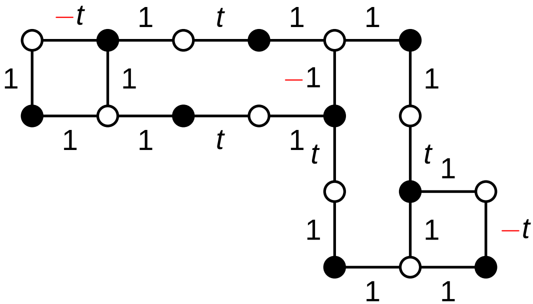
The Trefoil with a representation coming from a coloring.



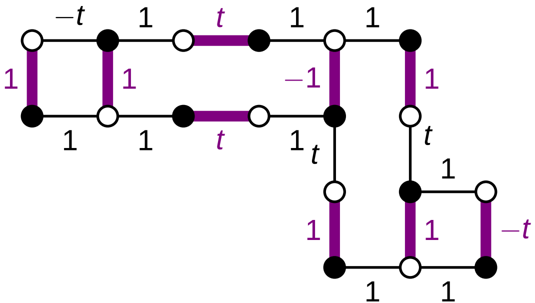
The bipartite graph untwisted into a plane graph.



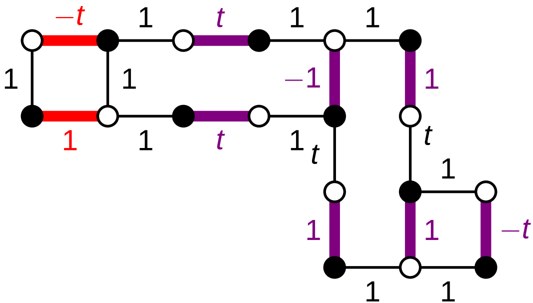
Labelling at each of the original crossings.



A Kasteleyn weighting.

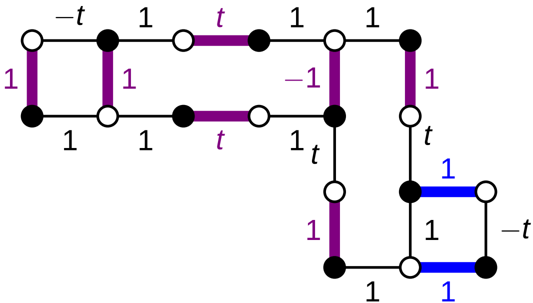


One perfect matching, yielding
$(1)(1)(t)(t)(-1)(1)(1)(1)(-t) = t^3$.



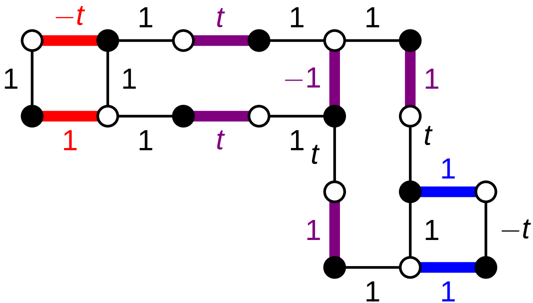
A second perfect matching, yielding

$(-t)(1)(t)(t)(-1)(1)(1)(1)(-t) = -t^4$.



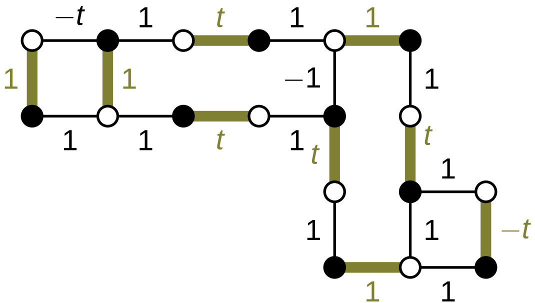
A third matching, yielding

$(1)(1)(t)(t)(-1)(1)(1)(1)(1) = -t^2$.



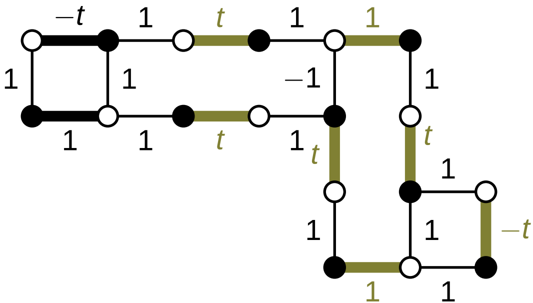
A fourth perfect matching, yielding

$(-t)(1)(t)(t)(-1)(1)(1)(1)(1) = t^3$.

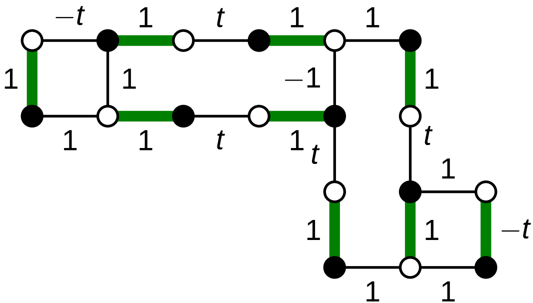


A fifth perfect matching, yielding

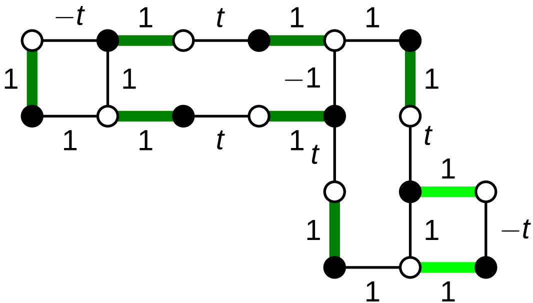
$(1)(1)(t)(t)(1)(t)(t)(1)(-t) = -t^5$.



A sixth perfect matching, yielding
$(-t)(1)(t)(t)(1)(t)(t)(1)(-t) = t^6$.

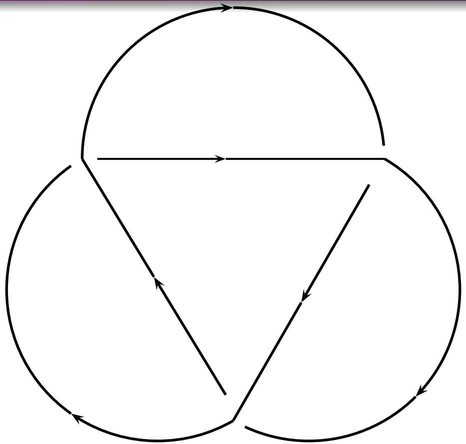


A seventh perfect matching, yielding
$(1)(1)(1)(1)(1)(1)(1)(1)(-t) = -t$.



An eighth perfect matching, yielding
$(1)(1)(1)(1)(1)(1)(1)(1)(1) = 1$.

Example: the Alexander Dimer for the Trefoil



The twisted Alexander polynomial is
$W_L(t) = t^6 - t^5 - t^4 + 2t^3 - t^2 - t + 1$.

Future directions

Open Question:

What can the combinatorics of Γ_ρ tell us about the representation ρ?

Conjecture:

Reducible representations give disconnected Γ_ρ.

Open Questions:

What can the study of random lifts of a graph Γ tell us about the possible representations ρ of the knot that gives Γ?

What else might a random lift model tell us here?