Despite working on several different topics over the years, my research theme has been the same: find useful tools from **combinatorics**, **graph theory**, and **matroid theory**, and employ them to attack questions in **low-dimensional topology** in subjects like **knot theory** and the study of **hyperplane arrangements**.

A *knot* is an embedding of the circle into three-space; because of its combinatorial description as a four-valent graph with crossing information called a *diagram*, a knot is is often a gateway for students into the study of three-manifolds. These can arise either as knot complements or from "surgery" on a knot diagram (a description of how to twist space).

Three-manifolds are not completely understood, despite several famous advances lately, including proofs of the Poincaré conjecture (Perelman 2002-3) and Virtually Haken conjecture (Agol 2012).

Knot theorists are still concerned with the main question of finding an invariant (or a collection of invariants) that can be computed relatively easily and can distinguish all knots.

A *hyperplane arrangement* is a finite collection of codimension one, linear, affine subspaces of a space; we restrict to lines in the plane. Real lines in the real plane can be completely understood via combinatorics because each line divides the space into two pieces. On the other hand, the complex setting is analogous to the real setting of two-dimensional subspaces in four-space, and this has obvious connections to open problems in algebraic geometry and low-dimensional topology.

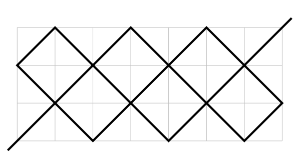
Main Question 1. What do large knots look like? What do invariants of large knots look like?

Knots are named by their minimal number of crossings, and we have tables of knots now up to sixteen crossings. Knot theorists have contributed to wiki-like pages by calculating our favorite invariants of these small knots.

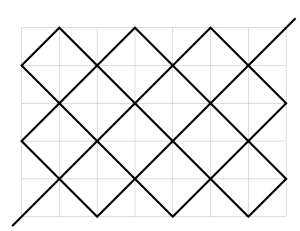
One would like to understand what happens to these invariants as the crossing number increases. Almost as a rule, the traditional way we attack knot theory is to create infinite families of knots and study their invariants. In this way, we can show that there exists a family of knots whose invariants grow like such and such, but this hardly gives us an idea of the general theory.

Our pattern-finding brains are actually obstructing the designing and understanding of large knots more generally. Instead, I propose using the *probabilistic method*, whose implementation in graph theory since the 1960's by Erdös has become a very successful endeavor. Only in the last ten or so years have we seen its utility in topology: the study of random 3-manifolds by Dunfield with W. Thurston [DT06b] and D. Thurston [DT06a]; random walks on the mapping class group in a recent work by Ito [Ito15]; and the Linial-Meshulam model [LM06] for random simplicial 2-complexes and its recent generalization by Costa and Farber [CF14] to k-complexes. Also see survey papers on the topology of random complexes by Kahle [Kah13] and Bobrowski and Kahle [BK14].

The difficulty as usual comes down to choosing a good combinatorial model for randomness. Specifically, we would like a model for which computations of knot invariants is reasonably efficient. I have been using a random model based on a model for all knots developed by Koseleff and Pecker:



A model for 2-bridge knots.



A model for 3-bridge knots.

One lofty goal is to acquire data to determine whether the following conjecture might hold.

Conjecture 2. [Jon00, Prob 1] *The Jones polynomial of a knot K is 1 $\Leftrightarrow$ K is the unknot.*

One might believe this to be true because Khovanov homology, a knot invariant of bigraded abelian groups whose graded Euler characteristic gives the Jones polynomial, has been shown by Kronheimer and Mrowka to detect the unknot [KM11]. Dasbach and Hougardy [DH97] found no counterexample in all knots up to seventeen crossings, including over two million prime knots, and this was extended to eighteen crossings by Yamada [Yam00]. Sikora and Tuzun [TS18, ST18] have just recently checked up to twenty-three crossings. Rolfsen with Anstee and Przytycki [APR89] and with Jones [JR94] sought a counterexample via mutations. Bigelow [Big02] outlined an algorithm to find a counterexample among four-strand braids.

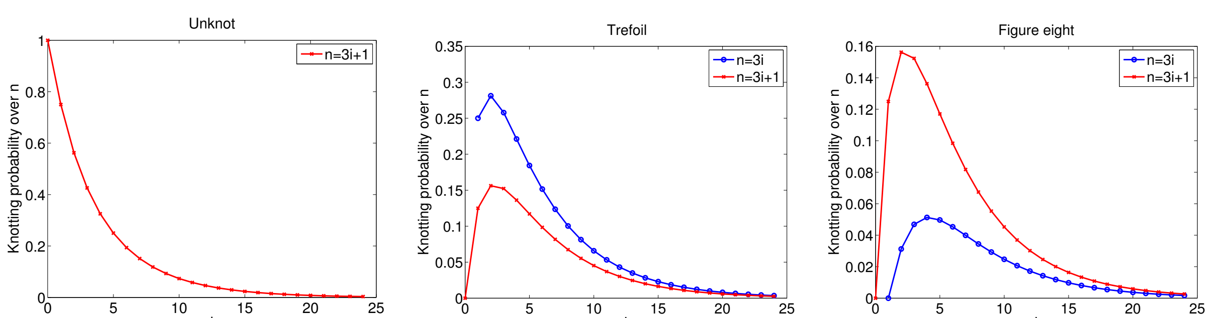
Ultimately, we would like to compare the probability of obtaining the unknot with that of obtaining unit Jones polynomial, but for this we need a useful notion of a space of knots. Various models for random knots from the literature, appearing more frequently since an AMS Special Session in Vancouver in 1993, generally take one of two forms: the physical knotting of a random walk or a random walk on the Cayley graph of the braid group.

Unfortunately, these models do not make common knot theoretic computations accessible. For this reason, we are interested in constructing a random model based on a particular knot diagram. The first paper on this topic appeared in 2014 by Even-Zohar, Hass, Linial, and Nowik [EZHLN16] using the übercrossing and petal projections of Adams et al. [ACD$^+$15]. Other random knot models can be found in recent work by Cantarella, Chapman and Mastin [CCM15], by Dunfield, Obeidin et al. [Dun14], by Even-Zohar [EZ17], and by Westenberger [Wes15].

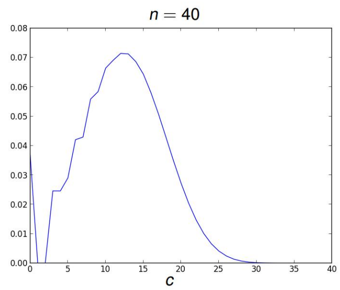
With Krishnan [CK15] and then with Even-Zohar and Krishnan [CEZK18], we apply the probabilistic method to a model developed by Koseleff and Pecker [KP16, KP11a, KP11b] that uses Chebyshev polynomials of the first kind to parametrize a knotted curve. These so-called *Chebyshev knots* are analogs of the more established Lissajous knots studied by Jones and Przytycki [JP98]. However, Koseleff and Pecker show in [KP11b] that all knots are Chebyshev, which is not the case for Lissajous knots (see for example work of Hoste with others [BDHZ09]).

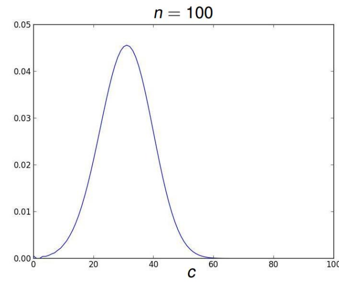
Chebyshev knots and links $T(a, b, c)$ can be projected to $(a-1) \times (b-1)$ billiard table diagrams like the ones pictured.The crossing information is determined by a string of $+$ and $-$ signs once a canonical ordering has been chosen. Koseleff and Pecker show that the class of k-bridge knots sits within the class of $(2k-1) \times b$ billiard tables; in particular, the $a = 3$ case contains all knots with bridge index at most two and the $a = 5$ case with that at most three.

Theorem 3. [CK15, CEZK18] *The probability of a given knot K appearing in $T(3, n+1)$ is given by a closed formula whose numerics are:*



Theorem 4. [CEZK18] *The probability of any knot with crossing number c in $T(3, n+1)$ is given by a closed formula whose numerics are:*





As n increases to infinity, one can imagine the initial slope smoothing in such a way that one would expect the following to hold:

Conjecture 5. *There is a Central Limit Theorem for the crossing numbers of knots in this model.*

Such a result, computed via the exact probabilities we have already obtained, or even proven theoretically, would revolutionize the subject. We would then turn to comparing various models for random knots so that such a Central Limit Theorem could extend to those models, as well.

This is work I am interested in pursuing once I have finished exploring my particular model, for which I still have many open questions:

Open Problem 6. Here are several possible directions:

(1) Obtain information about other invariants of random knots.
Explore conjectures made from experimental results of Dunfield et al [Dun14].
(2) Use my formulae in [Coh14] to compute the exact probability of obtaining unit Jones polynomial knots in $T(3, n)$.
(3) Use my formulae in [Coh14] to compute the exact probability of obtaining unit Jones polynomial knots in $T(5, n)$.
(4) Generalize a version of Schubert's Theorem (on the equivalence of 2-bridge knots) for 3- and higher-bridge knots, restricted just to the Chebyshev case, as it may not be true generally.
(5) Generalize some uniqueness results of Koseleff and Pecker that were used in [CK15].
(6) Obtain a formula for probabilities of knots in $T(5, n)$ analogous to that of Theorem 3.

In particular, problems (1), (4), and (5) have aspects that are suitable for **undergraduate research projects**, along with problems modeled after the game theoretic undergraduate research projects on knots discussed in an expository Math Horizons article with Henrich [CH12].

Another one of my research interests, complex projective line arrangements, has already proven to be appropriate for undergraduate research projects: in two years I've worked with six Vassar undergrads on four projects. I believe there to be many more projects available here.

Given the combinatorial intersection data of a line arrangement $\mathcal{A}$, we are interested in every possible geometric realization, and so we investigate the moduli space of $\mathcal{A}$, defined formally as $\mathcal{M}_{\mathcal{A}} = \{\mathcal{B} \in ((\mathbb{CP}^2)^*)^n | L(\mathcal{B}) = L(\mathcal{A})\}/PGL(3, \mathbb{C})$, where we mod out by the symmetries of $\mathbb{CP}^2$.

Main Question 7. How can we better understand $\mathcal{M}_{\mathcal{A}}$ for a line arrangement $\mathcal{A}$?

By Randell's Isotopy Theorem [Ran89], the embedding types of arrangements in the same connected component of $\mathcal{M}_{\mathcal{A}}$ are the same. Thus we understand arrangements for which $\mathcal{M}_{\mathcal{A}}$ is a single connected component. However, this is not the case in general; we have three well-known examples with disconnected $\mathcal{M}_{\mathcal{A}}$ due to MacLane, Falk and Sturmfels, and Nazir and Yoshinaga. Let us say that an arrangement is *exceptional* if it contains one of these three arrangements as a subarrangement.

Furthermore, Nazir and Yoshinaga [NY12] show that when $\mathcal{M}_{\mathcal{A}}$ is a single connected component and when a line is added to $\mathcal{A}$ in a trivial way, then the moduli space of the new arrangement is also a single connected component. Let us say that an arrangement is *reductive* if one of its lines was added in this trivial way: through at most two points of higher multiplicity.

A *Zariski pair* of line arrangements is a pair of lattice isomorphic arrangements $L(\mathcal{A})=L(\mathcal{B})$ with different embeddings in $\mathbb{CP}^2$. Some Zariski pairs are known: Rybnikov [Ryb11] found the first such pair (exceptional) in 1998 with 13 lines and showed furthermore that the complements have different fundamental groups; Artal Bartolo, Carmona Ruber, Cogolludo Agustín, and Marco Buzunáriz [ABRCAB05] give another example (exceptional and reductive) in 2003 with 11 lines and hint at a more general construction (exceptional); Guerville-Ballé [GB16] gives a newer example (reductive) in 2015 with 12 lines, shown later in the year to have complements with different fundamental groups by Artal Bartolo, Cogolludo Agustín, Guerville-Ballé, and Marco Buzunáriz.

Main Question 8. Are there unexceptional, nonreductive Zariski pairs?

There are no Zariski pairs of arrangements at all up to 9 lines [Fan97, GTV03, NY12, Ye13]. This led up to the following work on arrangements of 10 lines:

Theorem 9. [ATY13, ACTY13, ACS+15] *If there is a non-reductive, unexceptional Zariski pair of line arrangements of 10 lines, it must be on our list of fifteen arrangements.*

The classification of moduli spaces of arrangements of 11 lines was begun for those with quintuple or sextuple points [AGSX], and thirty-eight candidates for Zariski pairs have been identified. With undergraduate students at Vassar in two projects, I have continued this classification:

Theorem 10. [CL18] *Out of the four hundred ninety-five distinct arrangements obtained by adding an eleventh line through three or more double points in one of the ten* (10_3) *configurations, sixty-six of them has the moduli space necessary to be a Zariski pair of 11 lines.*

Theorem 11. [BCMS18] *An arrangement of 11 lines with some number of quadruple points and no points of higher multiplicity must contain one of fourteen subarrangements.*

Furthermore, there are thirty distinct non-reductive arrangements obtained from five of these fourteen subarrangements, and only one is a candidate for a Zariski pair.

One of my more general goals is to collect other potential Zariski pairs in order to compare their properties, and another is to develop further techniques to rule out arrangements from this list, as was successfully done in [ACS+15]. Following this, with my students we develop another new technique in [BCL+18] that has already ruled out some from their list.

Open Problem 12. Here are some more tractable projects:

(1) Generalize the result above of Nazir and Yoshinaga to more connected components and to higher dimensions.
(2) Apply this to rule out Zariski pairs of hyperplane arrangements, continuing work started by Gallet and Saini [GS16].
(3) Investigate other properties of 11-line arrangements found in [AGSX] following [ACS+15].

The purpose of these easier questions is to build a library of interesting cases from which we can make conjectures about more general behaviors. In particular, I am interested in the moduli space of *all* line arrangements (of up to n lines):

$$\mathcal{M}_n = \bigcup_{|\mathcal{A}|\leq n} \mathcal{M}_{\mathcal{A}}.$$

Main Question 13. How do the various $\mathcal{M}_{\mathcal{A}}$ interact together in $\mathcal{M}_n$?

Open Problem 14. Here are some less tractable projects:

(1) Impose metrics in $\mathcal{M}_n$ and study distances between $\mathcal{M}_{\mathcal{A}}$, as begun in [ACST15].
(2) Understand geodesics in this setting.

References

[ABRCAB05] Enrique Artal Bartolo, Jorge Carmona Ruber, José Ignacio Cogolludo-Agustín, and Miguel Marco Buzunáriz, *Topology and combinatorics of real line arrangements*, Compos. Math. **141** (2005), no. 6, 1578–1588.

[ACD+15] Colin Adams, Thomas Crawford, Benjamin DeMeo, Michael Landry, Alex Tong Lin, MurphyKate Montee, Seojung Park, Saraswathi Venkatesh, and Farrah Yhee, *Knot projections with a single multi-crossing*, J. Knot Theory Ramifications **24** (2015), no. 3, 1550011, 30.

[ACS+15] Meirav Amram, Moshe Cohen, Hao Sun, Mina Teicher, Fei Ye, and Anna Zarkh, *Combinatorial symmetry of line arrangements and applications*, Topol. Appl. **193** (2015), 226–247.

[ACST15] Meirav Amram, Moshe Cohen, Hao Sun, and Mina Teicher, *The distance between two real line arrangements along a parametrization and applications*, preprint available, 2015.

[ACTY13] Meirav Amram, Moshe Cohen, Mina Teicher, and Fei Ye, *Moduli spaces of ten-line arrangements with double and triple points*, arXiv:1306.6105, 2013.

[AGSX] Meirav Amram, Cheng Gong, Hao Sun, and Wan-Yuan Xu, *Moduli spaces of arrangements of 11 projective lines with a quintuple point*, Turk. J. Math., to appear.

[APR89] R. P. Anstee, J. H. Przytycki, and D. Rolfsen, *Knot polynomials and generalized mutation*, Topology Appl. **32** (1989), no. 3, 237–249.

[ATY13] Meirav Amram, Mina Teicher, and Fei Ye, *Moduli spaces of arrangements of 10 projective lines with quadruple points*, Adv. in Appl. Math. **51** (2013), no. 3, 392–418.

[BCL+18] Jordan Buhmann, Moshe Cohen, Baian Liu, Alexander May, and Shiyu Shu, *On transformations of the complex projective plane and applications to line arrangements*, 2018.

[BCMS18] Jordan Buhmann, Moshe Cohen, Alexander May, and Shiyu Shu, *Classification of moduli spaces of 11-line arrangements with quadruples*, 2018.

[BDHZ09] Adam Boocher, Jay Daigle, Jim Hoste, and Wenjing Zheng, *Sampling Lissajous and Fourier knots*, Experiment. Math. **18** (2009), no. 4, 481–497.

[Big02] Stephen J. Bigelow, *Does The Jones Polynomial Detect The Unknot?*, J. Knot Theory Ramifications **11** (2002), no. 4, 493–505.

[BK14] Omer Bobrowski and Matthew Kahle, *Topology of random geometric complexes: a survey*, arXiv:1409.4734, 2014.

[CCM15] Jason Cantarella, Harrison Chapman, and Matt Mastin, *Knot probabilities in random diagrams*, preprint arXiv:1512.05749, 2015.

[CEZK18] Moshe Cohen, Chaim Even-Zohar, and Sunder Ram Krishnan, *Crossing numbers of random two-bridge knots*, Topology Appl. **247** (2018), 100–114.

[CF14] Armindo Costa and Michael Farber, *Random simplicial complexes*, arXiv:1412.580, 2014.

[CH12] Moshe Cohen and Allison Henrich, *A knot game with _not KNerds*, Math Horizons **20** (2012), no. 2, 25–29.

[CK15] Moshe Cohen and Sunder Ram Krishnan, *Random knots using Chebyshev billiard table diagrams*, Topol. Appl. **194** (2015), 4–21.

[CL18] Moshe Cohen and Baian Liu, *Moduli spaces of one-line extensions of (10_3)-configurations*, preprint available, 2018.

[Coh14] Moshe Cohen, *The Jones polynomials of 3-bridge knots via Chebyshev knots and billiard table diagrams*, arXiv:1409.6614, 2014.

[DH97] Oliver T. Dasbach and Stefan Hougardy, *Does the Jones polynomial detect unknottedness?*, Experiment. Math. **6** (1997), no. 1, 51–56.

[DT06a] Nathan M. Dunfield and Dylan P. Thurston, *A random tunnel number one 3-manifold does not fiber over the circle*, Geom. Topol. **10** (2006), 2431–2499.

[DT06b] Nathan M. Dunfield and William P. Thurston, *Finite covers of random 3-manifolds*, Invent. Math. **166** (2006), no. 3, 457–521.

[Dun14] Nathan Dunfield, *Random knots: a preliminary report*, Slides for the talk available at http:// dunfield.info/preprints, 2014.

[EZ17] Chaim Even-Zohar, *The writhe of permutations and random framed knots*, Random Structures Algorithms **51** (2017), no. 1, 121–142.

[EZHLN16] Chaim Even-Zohar, Joel Hass, Nati Linial, and Tahl Nowik, *Invariants of random knots and links*, Discrete Comput. Geom. **56** (2016), no. 2, 274–314.

[Fan97] Kwai-Man Fan, *Direct product of free groups as the fundamental group of the complement of a union of lines*, Michigan Math. J. **44** (1997), no. 2, 283–291.

[GB16] Benoît Guerville-Ballé, *An arithmetic Zariski 4-tuple of twelve lines*, Geom. Topol. **20** (2016), no. 1, 537–553.

[GS16] Matteo Gallet and Elia Saini, *The diffeomorphism type of small hyperplane arrangements is combinatorially determined*, arXiv:1601.05705, 2016.

[GTV03] David Garber, Mina Teicher, and Uzi Vishne, π_1*-classification of real arrangements with up to eight lines*, Topology **42** (2003), no. 1, 265–289.

[Ito15] Tetsuya Ito, *On a structure of random open books and closed braids*, Proc. Japan Acad. Ser. A Math. Sci. **91** (2015), no. 10, 160–162.

[Jon00] V. F. R. Jones, *Ten problems*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 79–91.

[JP98] Vaughan F. R. Jones and Józef H. Przytycki, *Lissajous knots and billiard knots*, Knot theory (Warsaw, 1995), Banach Center Publ., vol. 42, Polish Acad. Sci. Inst. Math., Warsaw, 1998, pp. 145–163.

[JR94] Vaughan F. R. Jones and Dale P. O. Rolfsen, *A theorem regarding* 4*-braids and the* $V = 1$ *problem*, Proceedings of the Conference on Quantum Topology (Manhattan, KS, 1993), World Sci. Publ., River Edge, NJ, 1994, pp. 127–135.

[Kah13] Matthew Kahle, *Topology of random simplicial complexes: a survey*, arXiv:1301.7165, 2013.

[KM11] P. B. Kronheimer and T. S. Mrowka, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci. (2011), no. 113, 97–208.

[KP11a] P.-V. Koseleff and D. Pecker, *Chebyshev diagrams for two-bridge knots*, Geom. Dedicata **150** (2011), 405–425.

[KP11b] ———, *Chebyshev knots*, J. Knot Theory Ramifications **20** (2011), no. 4, 575–593.

[KP16] ———, *Harmonic knots*, J. Knot Theory Ramifications **25** (2016), no. 13, 1650074, 18.

[LM06] Nathan Linial and Roy Meshulam, *Homological connectivity of random 2-complexes*, Combinatorica **26** (2006), no. 4, 475–487.

[NY12] Shaheen Nazir and Masahiko Yoshinaga, *On the connectivity of the realization spaces of line arrangements*, Ann. Scuola Norm. Sup. Pisa **XI** (2012), no. 4, 921–937.

[Ran89] Richard Randell, *Lattice-isotopic arrangements are topologically isomorphic*, Proc. Amer. Math. Soc. **107** (1989), no. 2, 555–559.

[Ryb11] G. L. Rybnikov, *On the fundamental group of the complement of a complex hyperplane arrangement*, Funktsional. Anal. i Prilozhen. **45** (2011), no. 2, 71–85.

[ST18] Adam Sikora and Robert Tuzun, *Verification of the jones unknot conjecture up to 23 crossings*, arXiv:1809.02285, 2018.

[TS18] Robert E. Tuzun and Adam S. Sikora, *Verification of the Jones unknot conjecture up to 22 crossings*, J. Knot Theory Ramifications **27** (2018), no. 3, 1840009, 18.

[Wes15] Christopher David Westenberger, *Knots and Links from Random Projections*, ProQuest LLC, Ann Arbor, MI, 2015, Thesis (Ph.D.)–University of California, Davis. MR 3503478

[Yam00] S. Yamada, *How to find knots with unit Jones polynomials*, Knot Theory: Proceedings of the conference dedicated to Professor Kunio Murasugi for his 70th birthday, Toronto, July 13th-17th 1999 (M. Sakuma et al, ed.), Osaka University, 2000, pp. 355–361.

[Ye13] Fei Ye, *Classification of moduli spaces of arrangements of 9 projective lines*, Pacific J. of Math. **265** (2013), no. 1, 243–256.