Dimer Models for the Alexander and Twisted Alexander Polynomials of Knots

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Outline

1. The dimer model and matrices
   - Domino tilings and perfect matchings
   - Obtaining a determinant by a Kasteleyn weighting
   - Applying these ideas to knot theory

2. Dimer model for the Alexander polynomial
   - Sign Conventions
   - Example: the Trefoil

3. Twisted Dimer model for the twisted Alexander polynomial
   - Sign Conventions
   - Example: the Trefoil
   - Future directions
**Warmup**

**Question:** How many different ways can we tile a $2 \times n$ gameboard with $2 \times 1$ dominos?
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$n = 1$:

$n = 2$:

$n = 3$:


**Warmup**

**Question:** How many different ways can we tile a $2 \times n$ gameboard with $2 \times 1$ dominos?

$n = 1$

$n = 2$

$n = 3$
**Answer:** The number of ways $f(n)$ satisfies

$$f(n - 2) + f(n - 1) = f(n).$$
**Answer:** The number of ways $f(n)$ satisfies

$$f(n) = f(n - 2) + f(n - 1).$$

Thus we get $f(n)$ is the $n$-th Fibonacci number!
Combinatorialists ask how many ways to domino tile general gameboards.
Combinatorialists ask how many ways to domino tile general gameboards (and aztec diamonds).
Graph theorists look at the perfect matchings of associated bipartite graphs.
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A perfect matching $\mu$ is a collection of non-incident edges that covers the graph.
Physicists in statistical mechanics ask about the probability of achieving certain states.
From combinatorics to statistics

Physicists in statistical mechanics ask about the probability of achieving certain states.

A \textit{dimer} is just an edge.

A \textit{dimer covering} is a collection of non-incident dimers that covers the graph.
From combinatorics to statistics

Now let’s ask for the complete list of tilings and not just the number of them.
We use an idea from the generalization of the Matrix Tree Theorem.

So let’s look at matrices from the graphs.

Now let’s ask for the complete list of tilings and not just the number of them.
Consider the diagonal matrix $D$ where $d_{ij} = \deg v_i$, and let $A$ be the adjacency matrix of the graph.

**Matrix Tree Theorem (Kirchhoff):**

The determinant of any cofactor of the matrix $D - A$ gives the number of spanning trees of the graph.

The spanning trees themselves can be recovered by using variables to represent the edges in this matrix.
Matrices from graphs: bipartite adjacency matrix

Consider the adjacency matrix of the given bipartite graph $\Gamma$. 

\[
\begin{pmatrix}
0 & M \\
M^T & 0
\end{pmatrix}
\]

Call $M$ the **bipartite adjacency submatrix**.
Matrices from graphs: determinant and permanent

Recall the determinant of a matrix $M = (m_{ij})$

$$det(M) = \sum_{\sigma \in \mathcal{S}} \prod_{i} (-1)^{\text{sign}(\sigma)} m_{i\sigma(i)}$$

The **permanent** or **unsigned determinant** is

$$perm(M) = \sum_{\sigma \in \mathcal{S}} \prod_{i} m_{i\sigma(i)}$$
Proposition:

The terms in the permanent expansion of a bipartite adjacency submatrix associated with a bipartite graph give the complete list of perfect matchings of the graph.

Proof:

Each term in the permanent expansion is a permutation $\sigma$ matching each vertex $i$ in the first vertex set to a vertex $\sigma(i)$ in the second vertex set. □
Kasteleyn weighting: signing the edges

A **Kasteleyn weighting** of a plane bipartite graph is a signing of the edges such that \# negatives around a particular face is

- odd if the face has length 0 mod 4 or
- even if the face has length 2 mod 4.

**Lemma:**
Suppose $G$ has a Kasteleyn weighting. Then so does $G\setminus e$. 
**Proposition:**

The determinant expansion of a bipartite adjacency submatrix associated with a Kasteleyn-weighted bipartite graph gives the complete list of perfect matchings up to sign.

**Proof:**

Two permutations differ by a transposition $\leftrightarrow$

\[\exists \text{ four non-zero terms in a rectangle in the matrix} \leftrightarrow \exists \text{ a square face in the graph.}\]

\[\exists \! \text{ negative sign in each square, so these have opposite signs in both the matrix and the perfect matching.} \square\]
Main Results

**Theorem 1:**

Let $\Gamma$ be a balanced overlaid Tait graph associated with $K$. Then

$$\sum_m \prod_{e \in m} \mu(e) ≡ \Delta_K(t),$$

the Alexander polynomial of $K$ up to sign and a power of $t$.

(over all perfect matchings $m$ and all edges in each).
Main Results

**Theorem 2:**

Let $\Gamma_\rho$ be the $p$-lift obtained from $\Gamma$ for $K$ and $\rho$. Then

$$\sum_m \prod_{e \in m} \mu(e) \doteq W_K(t),$$

the twisted Alexander polynomial of $K$ with representation $\rho$ up to sign and a power of $t$

(over all perfect matchings $m$ and all edges in each).
A **knot** $K$ is $S^1$ embedded in $\mathbb{R}^3 \cup \infty$. We **orient** the knot.

A **knot diagram** $D$ is the projection of the knot onto $\mathbb{R}^2$ with under- and over-crossing information.

**Theorem (Reidemeister 1926):**

Two diagrams represent the same knot $\iff$  
$\exists$ a sequence of Reidemeister moves taking one to the other.
A knot invariant is an evaluation on a knot diagram that is constant under each of the three Reidemeister moves.
Why study knots?

**Understanding low-dimensional topology:**
- Topological quantum field theory gives 2-manifolds.
- Surgery on knots gives 3-manifolds.
- TQFT again gives 4-manifolds.
- Hyperbolic geometry triangulates knot complements.
- Number theory recognizes these hyperbolic volumes.

**By applying techniques from graph theory:**
- Need only study plane graphs!
- Nice properties emerge from the four-valent crossings.
- Tables of (small) knot diagrams can be found online.
A **signed graph** has edges weighted $+1$ or $-1$.

Checkerboard color the regions of a knot diagram $D$.

**Definition:**

The **signed Tait graph** $G$ associated with $D$ has

$V(G) = \{\text{colored regions}\}$ and $E(G) = \{\text{crossings of } D\}$.

Note that the dual $G^*$ comes from the uncolored regions.
Graphs from knots: the signed Tait graph $G$

Given a knot diagram,
Graphs from knots: the signed Tait graph $G$

checkerboard color the regions to get vertices
Graphs from knots: the signed Tait graph $G$

and sign the edges through the crossings
Graphs from knots: the signed Tait graph $G$.

to obtain the signed Tait graph $G$. 
Graphs from knots: the signed Tait graph $G$

Also use the white regions
Graphs from knots: the signed Tait graph $G$

to obtain the signed dual Tait graph $G^*$. 
Graphs from knots: the overlaid Tait graph $\hat{\Gamma}$

**Definition:**

The *overlaid Tait graph* $\hat{\Gamma}$ associated with $D$ is bipartite with $V(\hat{\Gamma}) = [E(G) \cap E(G^*)] \sqcup [V(G) \sqcup V(G^*)]$ and $E(\hat{\Gamma})$ the half-edges of $G$ and $G^*$.  

![Overlaid Tait Graph](image)

Each face in the overlaid Tait graph $\hat{\Gamma}$ is a square.
Graphs from knots: the overlaid Tait graph $\hat{\Gamma}$

Overlay the two Tait graphs
Graphs from knots: the overlaid Tait graph $\hat{\Gamma}$

to obtain the (unsigned) overlaid Tait graph $\hat{\Gamma}$.
**Definition:**

The **balanced overlaid Tait graph** $\Gamma$ associated with $D$ is obtained from $\hat{\Gamma}$ by removing two vertices from the larger set that lie on the same face:

```
  *  *
 /   \
|     |
  v   v
```

“Balanced” means the two vertex sets are the same size.
Graphs from knots: the balanced overlaid Tait graph $\Gamma$

Delete the starred vertices
Graphs from knots: the balanced overlaid Tait graph $\Gamma$

to obtain the balanced overlaid Tait graph $\Gamma$. 

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Dimer Models for Knots
The Alexander polynomial \( \Delta_K(t) \), a Laurent poly in \( \mathbb{Z}[t^{\pm \frac{1}{2}}] \), can be defined as the determinant of the Alexander matrix.

Compute the Fox partial derivatives of the relations with respect to the generators of the Dehn presentation for the fundamental group \( \pi_1(S^3 - K) \) of the knot complement.
After abelianization, this amounts to the following local information about each crossing:

Furthermore, Kauffman’s trick of assigning one minus sign at each crossing satisfies Kasteleyn’s weighting:
Example: the Alexander Dimer for the Trefoil

The bipartite graph for the Trefoil.
Example: the Alexander Dimer for the Trefoil

Labeling at each of the crossings.
Example: the Alexander Dimer for the Trefoil

Kauffman’s trick to yield a Kasteleyn weighting.

Kauffman’s trick to yield a Kasteleyn weighting.
Example: the Alexander Dimer for the Trefoil

One perfect matching, yielding \((t)(1)(t) = t^2\).
Example: the Alexander Dimer for the Trefoil

A second perfect matching, yielding \((t)(1)(-1) = -t\).
Example: the Alexander Dimer for the Trefoil

The third and last perfect matching, yielding \((1)(-1)(-1) = 1\).
The Alexander polynomial is $\Delta_L(t) = t^2 - t + 1$. 

Example: the Alexander Dimer for the Trefoil
The twisted Alexander polynomial $W_K(t)$, a poly in $R[t^{±1}]$, can be defined as the determinant of the Alexander matrix – after twisting by representation $\gamma : \pi_1(S^3 - K) \to GL_p(\mathbb{Z})$.

Replace each entry of the original Alexander matrix with a $p \times p$ block carrying the representation above.
We can use the following local information about each crossing:

![Diagram of crossing](image)

Replace each original vertex with $p$ copies, and replace each edge with $p$ edges, twisting as necessary.
Suppose the graph $\Gamma$ is obtained from a knot diagram, and let $\Gamma_\rho$ be obtained from $\Gamma$ together with the representation $\rho$.

**Definition:**
A $p$-**lift** of a graph on $n$ vertices is a graph on $pn$ vertices with a covering map.

**Proposition:**
By construction, $\Gamma_\rho$ is a $p$-lift of $\Gamma$. 
In order to use *Kauffman’s trick* as before, we need to take care of crossings coming from the “twists”.

We use Kuperberg’s *triple splitting* and *butterfly* tricks:

\[
\begin{align*}
&\quad w \\
&\quad w \\
&\quad v \\
&\quad v
\end{align*}
\]

\[
\begin{align*}
&\quad 1 \quad -1 \quad w \\
&\quad 1 \quad -1 \quad w \\
&\quad -1 \quad -1 \quad -vw
\end{align*}
\]
The Trefoil with a representation coming from a coloring.
The bipartite graph untwisted into a plane graph.
The dimer model and matrices
Dimer model for the Alexander polynomial
Twisted Dimer model for the twisted Alexander polynomial
Sign Conventions
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Future directions

Labelling at each of the original crossings.
The dimer model and matrices

Dimer model for the Alexander polynomial

Twisted Dimer model for the twisted Alexander polynomial

Sign Conventions

Example: the Trefoil

Future directions

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\(-t\)

\(t^1\)

\(-1\)

\(1\)

\(1\)

\(1\)

A Kasteleyn weighting.
One perfect matching, yielding
\[(1)(1)(t)(t)(-1)(1)(1)(1)(-t) = t^3.\]
A second perfect matching, yielding
\((-t)(1)(t)(t)(-1)(1)(1)(1)(-t) = -t^4\).
A third matching, yielding
\[(1)(1)(t)(t)(-1)(1)(1)(1)(1)(1)(1) = -t^2.\]
A fourth perfect matching, yielding
\((-t)(1)(t)(t)(-1)(1)(1)(1)(1)(1) = t^3\).
A fifth perfect matching, yielding
\[(1)(1)(t)(t)(1)(t)(t)(1)(-t) = -t^5.\]
A sixth perfect matching, yielding

\[ (-t)(1)(t)(t)(1)(t)(t)(1)(-t) = t^6. \]
A seventh perfect matching, yielding
\[(1)(1)(1)(1)(1)(1)(1)(1)(1)(-t) = -t.\]
An eighth perfect matching, yielding
The twisted Alexander polynomial is

\[ W_L(t) = t^6 - t^5 - t^4 + 2t^3 - t^2 - t + 1. \]
Future directions

Open Question:
What can the combinatorics of $\Gamma_\rho$ tell us about the representation $\rho$?

Conjecture:
Reducible representations give disconnected $\Gamma_\rho$.

Open Questions:
What can the study of random lifts of a graph $\Gamma$ tell us about the possible representations $\rho$ of the knot that gives $\Gamma$?
What else might a random lift model tell us here?